Robot Devices, Kinematics, Dynamics, and Control.

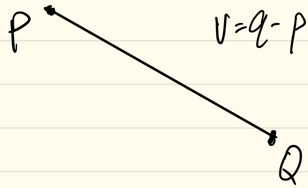
Office hour: Monday after class

Points & Vectors

Point: Location in 3D space

can subtract points
can't add them.

Vector: Distance & Direction



$v = q - p$

Representation of points & Vectors.

An array of n numbers

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \in \mathbb{R}^n$$ might represent a point or vector.

MLS (course Text) uses p, q for points (non-bold, Roman)

Bold face $\mathbf{v}, \mathbf{w}$ for vectors.

<u>Definition</u>: Linear Vector space is a set X and a field $\mathcal{F}$ together with two operations.

vector addition, scalar-vector multiplication.
Such that for $x, y, z \in X$, $\alpha, \beta \in \mathcal{F}$
$x + y \in X$ and $\alpha \cdot x \in \mathcal{F}$

closure

and the following actions are satisfied.

i) $x+y=y+x$

ii) $(x+y)+z = x+(y+z)$

iii) $\exists \vec{0} \in X$ such that $x+\vec{0}=x$

iv) $\alpha(x+y)=\alpha x+\alpha y$

v) $(\alpha+\beta) = \alpha x+\beta x$

vi) $(\alpha\beta)x=\alpha(\beta x)$

vii) $\underset{\uparrow F}{0} x = \underset{\uparrow X}{\vec{0}}$

$x+(-x)=0$

viii) $1 \cdot x = x$

Let V^n be normed n-dimentional real vector space.
An element of V^n is just of real numbers

$$V=\begin{bmatrix} V \\ \vdots \\ \vdots \\ V_n \end{bmatrix}, \quad V+W=[V_1+W_1, V_2+W_2, \cdots V_n+W_n]^T$$

A norm, $\|\cdot\|$, is a real-valued function. that satisfies.
for all $\alpha \in R$, $x \in V^n$

i) $\|\alpha x\| = |\alpha|\|x\|$

ii) $\|x+y\| \leq \|x\| + \|y\|$

iii) $\|x\| \geq 0$

(iv) if $\|x\|=0 \Rightarrow x=\vec{0}$

Example:

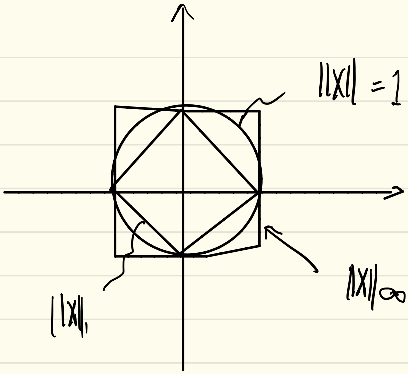
$$\|x\|_2 = \sqrt{x_1^2+x_2^2 \cdots x^2}$$

$$=\sqrt{x^T x}$$

$\|X\|_1 = |x_1| + |x_2| + \cdots |x_n|$

$\|X\|_\infty = \max |X|$

$\|X\|_p = \sqrt[p]{\sum x_i^p}$



Inner Product on V^n

$x, y \in V^n$

$\langle x, y \rangle = X^T y = x \cdot y$, $\|x\|_2 = \sqrt{x \cdot x}$

MLS uses R^n for real LVS

Physical vectors in 3D space can be represented by element V^n. but to do so, we have to establish a basis $\{i, j, k\}$.

If $\tilde{v}$ is a physical vector, then $v = \begin{vmatrix} \tilde{v} \cdot i \\ \tilde{v} \cdot j \\ \tilde{v} \cdot k \end{vmatrix}$

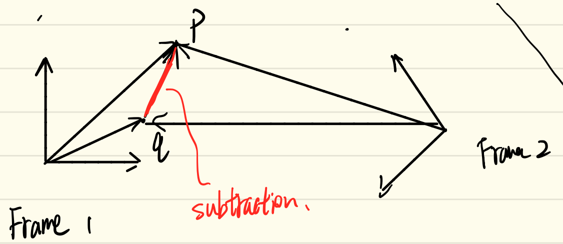
physical points can be represented . elements of $\mathbb{R}^3$ once we've established a frame $\{\theta, j, i, k\}$, where θ is the origin

so if $\tilde{p}$ is a physical point, we write:

$$P = \begin{bmatrix} (\tilde{p}-\theta)\cdot i \\ (\tilde{p}-\theta)\cdot j \\ (\tilde{p}-\theta)\cdot k \end{bmatrix}$$

A universal frame is implied / understood

If $p, q \in \mathbb{R}^3$ are points, then $v = p - q$ is a well-defined vector.



Points are not addable

$$V = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} \in V^n$$

Distance between points and Lengths of vectors

We will treat $\mathbb{R}^n$ as Euclidean space, with the standard two-norm used as a distance metric between points.

ie. $p, q \in \mathbb{R}^n$, $d(p,q) = \|p-q\|_2 = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$

Let $\theta \in \mathbb{R}^3$ denote the origin. frame, then $\sqrt{p_1^2+p_2^2+p_3^2}$ can only have meaning as $\|p - 0\|$. ie. distance to the origin.

Some Important Facts

$\forall$ = for all $\exists$ = there exists
$\exists!$ " " unique.

with the same field.

- Let X, Y be LVS's and let $f: X \to Y$. we say f is linear if it satisfied

(i) superposition
$\forall x_1, x_2 \in X$, $f(x_1 + x_2) = f(x_1) + f(x_2)$

(ii) scaling $\forall x \in X$
$f(\alpha x) = \alpha f(x)$ $\forall \alpha \in \mathbb{F}$

Theorem

Let X, Y be LVS over $\mathbb{F}$, and let $\alpha, \beta \in \mathbb{F}$ and $x_1, x_2 \in X$.

then a function $f: X \to Y$ is linear if and only if

$$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$$

Proof: (Both Direction)

Let $\alpha = \beta = 1$, $f(x_1 + x_2) = f(1 \cdot x_1 + 1 \cdot x_2) = 1 \cdot f(x_1) + 1\, f(x_2)$

$$= f(x_1) + f(x_2)$$

Proves (i) Superposition.

To prove (ii).

$$
\begin{aligned}
f(\alpha x_1) &= f(\alpha x_1 + \vec{0}) && \text{by axiom (3)} \\
&= f(\alpha x_1 + 0 \cdot x_2) && \text{by axiom (7) of LVS} \\
&= \alpha f(x_1) + 0 \cdot f(x_2) && (\beta = 0) \\
&= \alpha f(x_1) + \vec{0} && \text{by axiom 7} \\
&= \alpha f(x_1)
\end{aligned}
$$

"Only if"

$f(\alpha x_1 + \beta x_2) = f(\alpha x_1) + f(\beta x_2)$ axiom (i)

$= \alpha f(x_1) + \beta(x_2)$ by axion (ii) QED.

- A set of vectors $x_1 \cdots x_n \in X$ is said to be Linearly Dependant.

  if $\exists\, \alpha_1, \cdots, \alpha_n \in F$ (not all $\alpha = 0$), such that $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$

- A set of vectors is said to be Linearly Independant, if no such α's exist

Example:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2,$$ Linearly dependant since $\alpha_1 = 1, \alpha_2 = 1$, $\alpha_3 = -1$

Standard Basis in R^n

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$ are LI

Given $x_1, x_2 \in X$, the space of $\{x_1 \cdots x_n\}$ is the set of all vectors $V \in X$ of the form

$$V = \sum_{i=1}^{n} \alpha x_i \quad \text{for any scalers}, \ \alpha_1 \cdots \alpha_n$$

Related idea: colomn space of a matrix

$A = \begin{pmatrix} a_1 & \cdots & a_q \\ \vdots & & \vdots \end{pmatrix} \in R^{p \times q}$ where $a_i \in R^p$, colomn space $\underline{\text{space}\{a_1, \cdots a_q\}}$

A set of LI vectors $x_1, \cdots, x_n$ such that $X = \text{span}\{x_1, \cdots, x_n\}$ is a Basis for X.

Example
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \quad \text{where } e_i \in \mathbb{R}^n$$

is called the standard basis.

Consider the 2^{nd}-order Linear ODE

$$\ddot{x} + 3\dot{x} + 2x = 0 \qquad (*)$$

claim: real-valued solutions to $(*)$ form a LVS over $\mathbb{R}$.

Proof: Let $x_1(t), x_2(t), x_3(t)$ be solutions to $(*)$, and let $a, b \in \mathbb{R}$.

Closure

$x_1(t) + x_2(t)$ is also a solution. To check, plug into $(*)$.

$$\frac{d^2}{dt^2}(x_1 + x_2) + 3\frac{d}{dt}(x_1 + x_2) + 2(x_1 + x_2)$$

$$= \underbrace{(\ddot{x}_1 + 3\dot{x}_1 + 2x_1)}_{=0} + \underbrace{(\ddot{x}_2 + 3\dot{x}_2 + 2x_2)}_{=0}$$

$= 0$

$\Rightarrow$ closed under vector addition. Likewise ax_1 is a solution, seen by plugging into the LHS of $(*)$.

$\Rightarrow$ closed under scalar multiplication.

Defining $\vec{0} = 0(t)$, axioms all follow from perspective of $\mathbb{R}$

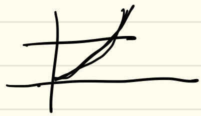
Need to show $\vec{0}$ is in the LVS. but obviously $\ddot{0} + 3\dot{0} + 2 0 = 0$.

By facts of differential equations, all solutions to $(*)$ can be expressed as linear combination of $V_1(t) = e^{-2t}$, $V_2(t) = e^{-t}$

Call the solution to $(*)$ X. Clearly, $X = \text{span}\{X_1(t), X_2(t)\}$

Must also show $v_1(t)$ and $v_2(t)$ are LI to prove they are basis.

$$a_1 e^{-t} + a_2 e^{-2t} = 0 \quad \Rightarrow \quad a_1 e^{-t} = -a_2 e^{-2t}$$



$$-a_1 e^{t} = a_2$$

can only when $a_1 = a_2 = 0$ for all t.

Now we can represent all members of X w.r.t $\{V_1(t), V_2(t)\}$

$$X(t) = a_1 V_1(t) + a_2 V_2(t) = [V_1(t) \quad V_2(t)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$y(t) = b_1 V_1(t) + b_2 V_2(t) = [V_1(t) \quad V_2(t)] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is the representation of x w.r.t $\{V_1(t), V_2(t)\}$

and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ . . . .

- Choice of basis is not unique.
- However, the number of basis elements is constant (for a given LVS). Call the number n, then we say we have an "n-dimensional LVS"

MATRIX Facts.

- A square matrix is said to be singular if $\det M = 0$
- The column of a non-singular matrix and LI $\Rightarrow$ they span $\mathbb{R}^n$
- the rows are also LI
- A matrix $M \in \mathbb{R}^{n\times n}$ is said to be SKEW SYMMETRIC if $M = -M^T$. example: $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- $M \in \mathbb{R}^{n\times n}$ is SYMMETRIC if $M = M^T$

  eg $M = \begin{bmatrix} 1 & -3 \\ -3 & 7 \end{bmatrix}$
- The cross product of two vectors. $x, y \in \mathbb{R}^3$

$$x \times y = \underbrace{\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}}_{\hat{x}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -x_3 y_2 + x_2 y_3 \\ x_3 y_1 - x_1 y_3 \\ -x_2 y_1 + x_1 y_2 \end{bmatrix}$$

Also $x \times y = -y \times x = -\hat{y}x$

Note: $x \times y$ is Bilinear

Consider $x \in \mathbb{R}^3$. what are the eigen values of $\hat{x}$?

Direct Computation:

$$\det(\lambda I - \hat{x}) = 0$$

$$= \lambda(\lambda^2 + \|x\|^2) = 0$$

$$\Rightarrow \lambda = 0, \pm j\|x\|$$

Rigid Transformations (Follow MLS)

- Let $p(t) \in \mathbb{R}^3$ be a function of time, intepreted as a moving point.

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \in \mathbb{R}^3$$

Likewise for $q(t) \in \mathbb{R}^3$. a rigid body is one that for any two points $p(t)$, $q(t)$

$d(p(t), q(t)) = \|$ Dif ...

- A ri... motion ... points satisfies

- A rigid displacement is the net movement of a rigid body from one location to another under rigid motion.
- Let $g = \mathbb{R}^3 \to \mathbb{R}^3$ mapping of points.

Natural way to define a corresponding actions on vectors.

Pick an arbitary base point $p_0 \in \mathbb{R}^3$

define $g_* : V^3 \to V^3$ via $g_*(v) = g(p_0 + v) - g(p_0)$

- A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid displacement if
  1. length is preserved $\|g(p) - g(q)\| = \|p - q\|$
  2. Cross product of vectors is preserved in this sense:

     $g_*(v \times w) = g_*(v) \times g_*(w)$

     $\forall v, w \in V^3$ (orientation preserving)

Facts about rigid displacement.

- The induced action g^* of a rigid displacement is linear $g_*(\alpha v + \beta w) = \alpha g_*(v) + \beta g_*(w)$

- Inner products are preserved. $v \cdot w = \frac{1}{4}(\|v+w\|^2 - \|v-w\|^2)$

  polarization identity.

$= \frac{1}{4}(\|g_*(v+w)\|^2 - \|g_*(v-w)\|^2) \Rightarrow g_*$ is length preserving since g is preserving.

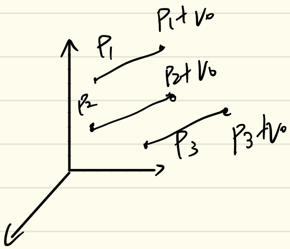
$= \frac{1}{4}(\|g_*(v)+g_*(w)\|^2 - \|g_*(v)-g_*(w)\|^2)$ — linearity of g_*

$= g_*(v) \cdot g_*(w)$

polarization un-done!

Translations.

$g(p) = p + v_0$. where $v_0 \in V^2$ is a (pure) translation.



Rotations :

Such that

Define $SO(n) = \{ R \in \mathbb{R}^{n\times n} : \underbrace{R^T R = I}_{\text{distance preserving}}, \underbrace{\det R = +1}_{\text{orientation preserving}} \}$

SO : is the set of "special Orthogonal" matrices.

Claim : Elements of $SO(n)$ are rigid transformations.

Proof : Let $p, q \in \mathbb{R}^n$. let $g(p) = Rp$, then g preserves length.

$$\|Rp - Rq\| = \|R(p-q)\| = \sqrt{(R(p-q))^T (R(p-q))}$$

$$\sqrt{(p-q)^T \underbrace{R^T R}_{I} (p-q)} = \sqrt{(p-q)^T I (p-q)} = \sqrt{(p-q)^T (p-q)} = \|p-q\|$$

$$g_*(v) = g(p+v) - g(p) = R(p+v) - Rp = Rv$$

Claims : $R(v \times w) = (Rv) \times (Rw)$

Pf : P.S 1.

- General Displacements in 3dimensions

Given $R \in SO(3)$, $v \in V^3$

Define $g(p) = Rp + v$

claim : g is a rigid transformation.

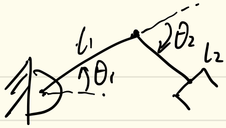
Overview of 530.646

1) Kinematics.
   - Forward
   - Inverse
   - Differential

2) Dynamics

3) Control
   - of serial link
   - fully actuated
   - robot manipulator.

Example to cover (1). (2). (3)



$\circlearrowleft$ CCW

Description: Planar, two-link solid, revolute manipulator

1) Kinematics.

- Foward. Position / orientation of end-effection as a function of as a function of θ_1, θ_2

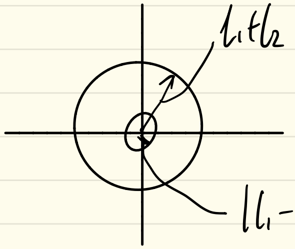
Kinematic map

$$f: T^2 \to \underbrace{SE(2)}_{\to SO(2)\times\mathbb{R}^2} \quad \text{or simply but slightly inaccurate:}$$

$$f: \mathbb{R}^2 \to \mathbb{R}^3 \qquad f: \begin{vmatrix}\theta_1\\ \theta_2\end{vmatrix} \to \begin{vmatrix}x\\ y\\ z\end{vmatrix}$$

$$f(\theta_1, \theta_2) = \begin{vmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{vmatrix} \Big\} \bar{f}(\theta_1, \theta_2)$$

Also part of kinematics



(b) Inverse Kinematics.

Given end- "pose" → position + orientation

find θ_1, θ_2

these since dim of co-domin is $\underline{3}$
but the dim of domain is $\underline{2}$
The problem is ill-posed.

so let's consider

$$f = \mathbb{R}^2 \rightarrow \mathbb{R}^3 \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \bar{f}(\theta_1, \theta_2) = \begin{bmatrix} l_1 \cos \\ 0 \end{bmatrix}$$

(c) Differential Kinematics

Idea: compute $\begin{vmatrix} \dot{x} \\ \dot{y} \end{vmatrix}$ given θ_1. $\dot{\theta}$ using the chain rule.

$$\begin{vmatrix} x \\ y \end{vmatrix} = \bar{f}(\theta_1, \theta_2)$$

$$\Rightarrow \begin{vmatrix} \dot{x} \\ \dot{y} \end{vmatrix} = \begin{vmatrix} \frac{\partial \bar{f}}{\partial \theta_1} \cdot \dot{\theta}_1 + \frac{\partial \bar{f}_1}{\partial \theta_2} \dot{\theta}_2 \\ \frac{\partial \bar{f}_2}{\partial \theta_1} \cdot \theta_1 + \frac{\partial \bar{f}_2}{\partial \theta_2} \cdot \dot{\theta}_2 \end{vmatrix} = \overbrace{\begin{bmatrix} \frac{\partial \bar{f}_1}{\partial \theta_1} & \frac{\partial \bar{f}_1}{\partial \theta_2} \\ \frac{\partial \bar{f}_2}{\partial \theta_1} & \frac{\partial \bar{f}_2}{\partial \theta_2} \end{bmatrix}}^{\text{Jacobian Matrix}} \begin{vmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{vmatrix}$$

$$\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} \bar{f}_1(\theta_1, \theta_2) \\ \bar{f}_2(\theta_1, \theta_2) \end{vmatrix} \qquad \bar{f}_1(\theta) = l_1 \cos\theta_1 + l_2$$

$$\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \bar{x} = \bar{f}(\theta) \qquad \Rightarrow \dot{\bar{x}} = J(\theta) \cdot \dot{\theta}$$

$$\dot{\bar{x}} = \frac{\partial \bar{f}}{\partial \theta} \cdot \dot{\theta}$$

2) Dynamics

Larg... Equation

3) Controls

Group

Def: A group is a set $\mathcal{Y}$ together with a binary operator

$\underline{f: \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}}$ that satisfied

closed

for all $a, b, c \in \mathcal{Y}$

1. closure: $f(a,b) \in \mathcal{Y}$

2. assoc: $f(a, f(b,c)) = f(f(a,b), c)$

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3. $\exists e \in \mathcal{Y}$ st. $ae = a$

4. For each $a \in \mathcal{Y}$, $\exists a^{-1}$ st $a \cdot a^{-1} = e$

Recap

i) Kinematics

FWD. f: "Joint Space" $\rightarrow$ "End-effector", configuration space"

$f: Q \Rightarrow SE(n)$, $n=2,3$

$\leftarrow Q =$ Joint Space

Example

$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ Then conside - $\tilde{f}: \begin{vmatrix} \theta_1 \\ \theta_2 \end{vmatrix} \Rightarrow \begin{vmatrix} x \\ y \end{vmatrix}$

$f: \begin{vmatrix} \theta_1 \\ \theta_2 \end{vmatrix} \rightarrow \begin{vmatrix} x \\ y \\ z \end{vmatrix}$ Inv ...

- Differential Ki...

$$\dot{x} = \underbrace{J(\theta)}_{\frac{\partial f}{\partial \theta}} \dot{\theta}$$

supposed I want $\dot{x} = V^*$, a desired velocity, knowing θ, and being able to command $\dot{\theta}$

Then set $\dot{\theta} = J^{-1}(\theta)\, V^*$ (J^{-1}: matrix inv.)

Inverting J requires L.I. of columns.

Group

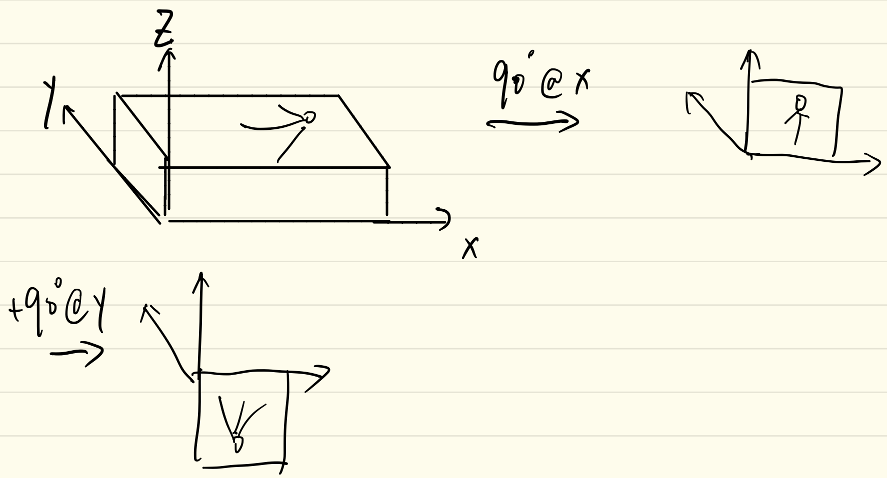
Def: a set and a binary operator, $\circ: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (closed)

that satisfies

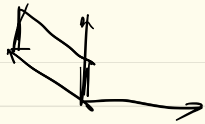
1. Associtivity. $\forall a, b, c \in \mathcal{G}$, $a \cdot (b \cdot c) = (a \cdot b) c$
2. $\exists e \in \mathcal{G}$, s.t. $a \cdot e = a$. $\forall a \in \mathcal{G}$

   (Also, $e \cdot a = a$, can be proved)
3. For each a in $\mathcal{G}$. $\exists a^{-1} \in \mathcal{G}$, st $a \cdot a^{-1} = e$

   (also true: $a^{-1} \cdot a = e$)



$90^\circ @ Y \rightarrow$ $90^\circ @ X \rightarrow$



Representing 3D Rotations

- Rotation matricies , $R \in SO(3)$

Plusses . • Sequential Rotations. R_1, R_2, R_3 are easily combined via matrix multiplication to compute net rotations.

If R_1. then R_2.

$$x \rightarrow (R_1 x)$$

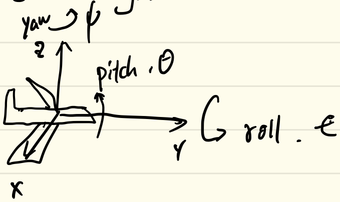
$$(R_1 x) \rightarrow R_2 (R_1 x)$$

$$\Rightarrow x \rightarrow (R_2 . R_1) x$$

• Inverse is easy $R^{-1} = R^T$

Minuses : 9 numbers of 3 DOF

Euler angles



$R_y(\theta)$ = Rotation about y. angles θ

$R_x(\phi)$ = . . . x . ϕ

$R_z(\psi)$ = . — — z

$$R = R_z R_y R_x$$

Pluses: 3 numbers for 3 DOF

Minuses: Singularities.

* Look up [illegible].

Representing Rotations (Recap)

- $SO(3) \rightsquigarrow$
  - 9 numbers for 3 DOF
  - No singularities
  - Group easy to composite.
- Euler Angle $\rightsquigarrow$
  - 3 numbers for 3DOF
  - small angles
  - singularities
- unit Quterinian $\rightsquigarrow$ 4 variable number slightly redundant.

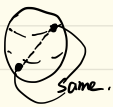
$Q = (\alpha, \vec{q})$, where $\alpha \in R$, $\vec{q} \in R^3$

require $\alpha^2 + \|\vec{q}\| = 1$

$Q = (\alpha, \vec{q}) = (\cos\frac{\theta}{2}, w\sin\frac{\theta}{2})$, where $w \in R^3$, with $\|w\| = 1$, $\theta \in \mathbb{R}$

$\|Q\| = 1$

Note: $SO(3) = S^3 / p \sim -p$



Another way to write Quaterion. : $a+ib+jc+kd$ $i^2=1$. not commutative.

Def : A group G with its operator $\circ$, acts on a space X. If there is a mapping $f: G \times X \to X$ satisfying :

1. identity action : if $e \in G$ is the identity element in G $f(e,x)=x$, $\forall x \in X$

2. Associativity : $\forall g, h \in H$ and $x \in X$, then $f(g, f(h,x)) = f(g \circ h, x)$

Example

Translation Group

Recall V^n is LVS $\Rightarrow$ V^n is a group under addition

We can say V^n acts on points in R^n in the usual way: vector, point addition.

Proof: $G = V^n$. $X = R^n$, here $f = \underbrace{\text{"}+\text{"}}_{\text{action}}$, also $\circ = \text{"}+\text{"}$ ← Group operator.

Closure: $\forall v, w \in V^n$, $P \in R^n$. $v+P \in R^n$. by properties of point-vector addition.

1. $e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is identity element. of V^n and $e+p=p$, $\forall p \in R^n$, obviously.

2. Associativity . $\underbrace{V}_{}\underset{\text{action}}{+}(w\underset{\text{action}}{+}p) = (v\underset{\text{group operator.}}{+}w)\underset{\text{action}}{+}p$ by properitis of vector, point-vector addition.

Example : $(R, +)$ Group.

$X = R$

candidate action $f(a, x) = ax + x + 1$

$a \in G$ ↑ ↖ R

Fails both axioms.

1) $e = 0$, $f(e, x) = x + 1 \neq x$

2) $a, b \in G$, $x \in X$. $f(a, f(b, x)) = a + b + x + 2 \neq f(a \circ b, x) = a + b + x + 1$

Group Products

Given two groups $(G, \circ)$ and $(H, \diamond)$

<u>Direct Product</u> : Given $g_1, g_2 \in G$, $h_1, h_2 \in H$

Then $(g_1, h_1) \in G \times H$

$(g_2, h_2) \in G \times H$

And the direct group product would be

$$(g_1, h_1) \square (g_2, h_2) = (g_1 \circ g_2, h_1 \diamond h_2)$$

Example : $G = SO(3)$ (matrix multiplication)

$H = V^3$ (vector addition)

Direct Product : $G \times H$ if $(R_1, V_1), (R_2, V_2) \in G \times H$

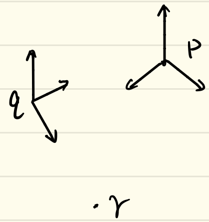
then $(R_1, V_1) \square (R_2, V_2) = (R_1 R_2, V_1 + V_2)$

Consider $SE(3)$ acting on R^3. $G = (R, V) \in SE(3)$

we know $g(P) = Rp + v$

Example: $90°$ Yaw, move 1-m along x-axis.

$$R_1 = \begin{vmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad v_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \qquad p = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$



$$R_1 p + V_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

$$R_2 = I, \quad V_2 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \quad r = R_2 q + V_2 = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix}$$

Apply in another order

$$R_2 p + V_2 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = q$$

$$R_1 q + V_1 = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}$$

Doesn't match $R_1 R_2 p + V_1 + V_2 = \begin{vmatrix} 2 \\ 0 \\ 0 \end{vmatrix}$

Roll → Pitch → Yaw   world

Yaw → Pitch → Roll   "Nested." frames.

Fix the model for how to combine groups to model rigid transformation.

Apply (R_2, V_2) to p, then apply (R_1, V_1) to the result.

$$g_2(p) = R_2 p + V_2 = q \qquad g_1(P) = R_1 q + V_1 = R_1(R_2 p + V_2) + V_1$$

$$= R_1 R_2 p + \underbrace{(R_1 V_2 + V_1)}_{!!! \; \neq V_1 + V_2}$$

Semi-direct group product

$$g_1 \cdot g_2 = (R_1 R_2 \ , R_1 V_2 + V_1)$$

(R_1, V_1) (R_2, V_2)

Works B/C $SO(3)$ acts on V^3

Homogeneous Representation of points. vectors & rigid transformation.

Points $p = \begin{vmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{vmatrix}$ vectors $V = \begin{vmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{vmatrix}$

$$p+q = \begin{vmatrix} p_1 + q_1 \\ \vdots \\ p_3 + q_3 \\ 2 \end{vmatrix}$$ Not a point. Not allowed

$$p + V = \begin{vmatrix} p_1 + V_1 \\ p_2 + V_2 \\ p_3 + V_3 \\ 1+0 \end{vmatrix}$$ is a point

$$\bar{v} + \bar{w} = \begin{vmatrix} V_1 + W_1 \\ \vdots \\ 0 \end{vmatrix}$$ is a vector

Rigid Transformations

$$\bar{g} = \begin{vmatrix} R & V \\ \underbrace{0^T}_{1\times 3} & 1 \end{vmatrix}$$

In this Rotation

$$\bar{g}\bar{P} = \begin{vmatrix} R & V \\ 0^T & 1 \end{vmatrix} \begin{vmatrix} P \\ 1 \end{vmatrix} = \begin{vmatrix} RP + V \\ 1 \end{vmatrix}$$

$$\bar{g}\bar{w} = \begin{vmatrix} R & V \\ 0 & 1 \end{vmatrix} \begin{vmatrix} w \\ 0 \end{vmatrix} = \begin{vmatrix} Rw \\ 0 \end{vmatrix} = \begin{vmatrix} g_*(w) \\ 0 \end{vmatrix}$$

Usually abuse notation and write homogeneous representation without "$-$"

$$\Rightarrow \quad gp = (Rp + v) = \begin{vmatrix} R & v \\ 0^T & 1 \end{vmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{vmatrix} R & v \\ 0 & 1 \end{vmatrix} p$$

Recap.

- Basic Math Def...
  - LVS , linearity
  - Euclidean Space.
  - Points vs. Vectors
  - Groups
  - Group Action

- Rigid Transformations
  - Def n
  - Two key examples.
    - Translations $\mathbb{R}^3$
    - Rotations $SO(3)$
  - Semi-direct Product : $SE(3)$

    $SE(3) = SO(3) \circledS \mathbb{R}^3$

    special Euclidean Group

    semi-direct product.

  - Homogeneous Representation.

$$\sigma : (R, v) \longmapsto \underbrace{\begin{bmatrix} R & v \\ 0^T & 1 \end{bmatrix}}_{g}$$

Homomorphism

$\Rightarrow R = g(1:3, 1:3)$

$\Rightarrow v = g(1:3, 4)$

$p \mapsto \begin{vmatrix} p \\ 1 \end{vmatrix}$ homo point

$v \mapsto \begin{vmatrix} v \\ 0 \end{vmatrix}$ homo vector.

$Rp + v = gp$

- Representing Rotation
  - (Represent Translation : 3-vector)
  - Euler Angles
  - Quaterinions , $S^3/\mathbb{RP}$
  - Axis angle. $\langle \omega, \theta \rangle$, $\omega \in \mathbb{R}^3$, $\|\omega\| = 1$, $\theta \in \text{Rad} . 2\pi$

Degree of Freedom

- (Locally) The dimension of the configuration space.
- minimum # of real parameters to coordinate / param... the space.

Given a set of n parameters subject to a set of n constraint equations.
(Assume $m \leq n$)

so, $s_1 \cdots s_n$ subject to f

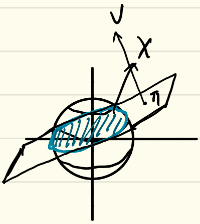
If rank $J = m$, then the constraints are independent. and there are $r = n - m$ DOF.

In general, there are $r = n - \text{rank}(J)$ DOF

Consider $S^2 = \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \}$ $\quad x^T x = 1$

$n = 3, \quad m = 1, \quad f(x_1, x_2, x_3) = x^T x - 1$

$J = [2x_1 \ \ 2x_2 \ \ 2x_3]$ which is always rank 1 on S^2, $r = 3 - 1 = 2$ DOF



$\pi = \{ x \in \mathbb{R}^3 : x^T v = 0 \}$

$\mathcal{X} = S^2 \cap \pi = \{ x \in \mathbb{R}^3 : \underbrace{x^T x - 1 = 0}_{f_1(x)}, \ \underbrace{x^T v = 0}_{f_2(x)} \}$

$J = \begin{bmatrix} 2x^T \\ v^T \end{bmatrix}$

x cannot be parallel (and therefore L.D upon) v!

ie. rank$(J) = 2$ on $\mathcal{X}$

Matrix Exponentials

Exponential Review:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ solves among other things how to deal with irrational exponents.

Also

$e^{i\theta} = \cos\theta + i\sin\theta$

Exponential function (or "map") is smooth and is the eign function of differentiation.

$$\boxed{\frac{d}{dx}} e^x = e^x \qquad \frac{d}{dx} \underbrace{e^{ax}}_{\text{eign function}} = \overset{\text{eign value.}}{a} e^{ax}$$

linear operator.

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \in \mathbb{R}^{n\times n}$$

where $A \in \mathbb{R}^{n\times n}$, $t \in \mathbb{R}$ $(A \in \mathbb{R}^{n\times n})$

Look at $\frac{d}{dt} e^{At} = \frac{d}{dt}(I + At + \frac{1}{2!}A^2t^2 \cdots)$

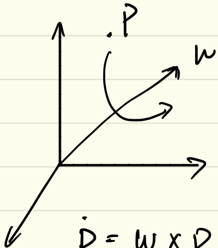
$$= 0_{n\times n} + A + A^2t + \frac{1}{2!}A^3t^2 + \frac{1}{3!}A^4t^3 + \cdots$$

$$= A(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots)$$

$$= Ae^{At}$$

Could have factored out and to the right $\Rightarrow \frac{d}{dt}e^{At} = Ae^{At} = e^{At}\cdot A$

In general $e^{At}\cdot B \neq B\cdot e^{At}$



w : constant

$\dot{P} = w \times P$

$= \hat{w} P$

Claim $P_0(t) = e^{\hat{w}t} P_0$ where P_0 is the initial position.

Proof: $\frac{d}{dt} P(t) = \frac{d}{dt}(e^{\hat{w}t} P_0)$

$$= \frac{d}{dt}(e^{\hat{w}t}) P_0$$

$$= \hat{w} e^{\hat{w}t} P_0$$

$$= \hat{w} P(t) \Rightarrow \text{which satisfy}$$

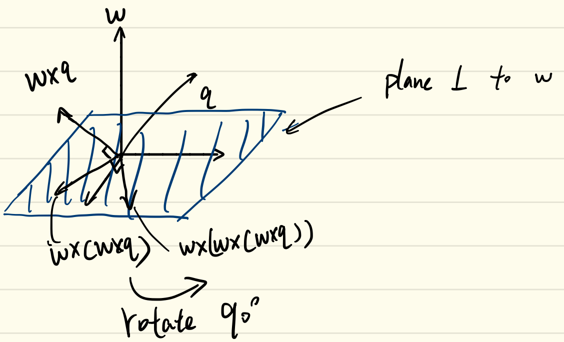
$$\dot{P} = \hat{w} P$$

$e^{\hat{w}t}$ has a lot of structures.

$e^{\hat{w}t} \cdot q$, q is a point , $t = 1$

$= q + \hat{w} q + \frac{1}{2!} \hat{w}^2 q + \frac{1}{3!} \hat{w}^3 q + \frac{1}{4!} \hat{w}^4 q + \cdots$

Look at $\hat{w}^k q = (\cdots (w \times (w \times (w \times q))))$



$\hat{w}^3 q = \|w\|^2 (-\hat{w} q)$

$= -\|w\|^3 \hat{w} q$

In general k odd $\hat{w}^k = \pm \|w\|^{k-1} \hat{w}$

alternating sign.

k even $\hat{w}^k = \pm \|w\|^{k-1} \hat{w}^2$

Group terms

$e^{\hat{w}} = I + (\sin\theta) \hat{w} + (1 - \cos\theta) \hat{w}^2$

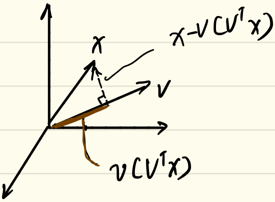
$\theta = \|w\|$

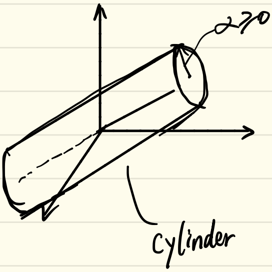
n parameters m constraints   "independent"

$r = n - m = \text{DOF}$

Example : $\mathcal{X} = \{ x \in \mathbb{R}^3 : \underbrace{\|x - vv^T x\|^2}_{\|(I - vv^T)x\|^2} = 0 \}$

$\|v\| = 1$





$$\mathcal{X}_\alpha = \{ x \in \mathbb{R}^3 : \underbrace{\|(I - vv^T)x\|^2 - \alpha^2}_{m = 1 \text{ constraint}} = 0 \}$$

$r = n - m = 3 - 1 = 2$ DOF

Constraint Function

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

symmetric, doesn't matter.

$$f(x) = x^T \underbrace{(I - vv^T)^T (I - vv^T)}_{(I - vv^T)} x - \alpha^2$$

$$(I - vv^T)(I - vv^T) = I - 2vv^T + vv^T vv^T = I - vv^T$$

$$f(x) = x^T (I - vv^T) x - \alpha^2$$

Fact: $\frac{\partial}{\partial x}(x^T P x) = 2x^T P$. P is symmetric.

$$\Rightarrow \frac{\partial f}{\partial x} = 2\overbrace{x^T(I - vv^T)}^{\neq 0} \in \mathbb{R}^3$$

$$\text{rank}\left(\frac{\partial f}{\partial x}\right) = 1 \quad \text{if } \frac{\partial f}{\partial x} \neq 0$$

$$\left\|\left(\frac{\partial f}{\partial x}\right)^T\right\|^2 = 4\|(I - vv^T)x\|^2 = \alpha^2 > 0$$

when $\alpha = 0$, $f(x) = x^T(I - vv^T)x$

$$\frac{\partial f(x)}{\partial x} = 2x^T(I - vv^T) = [0 \quad 0 \quad 0]$$

rank d find, and $\therefore$ can not apply counting argument

Rank is a robust property

An alternative de of X

$$X = \{x \in \mathbb{R}^3 : \underbrace{(I - vv^T)x}_{f(x)} = 0\}$$

$$\frac{\partial f}{\partial x} = (I - vv^T)$$

$\text{rank}(I - vv^T) = 2$ $\qquad r = n - 2 = 3 - 2 = 1$

Derivative Notation

$f : \mathbb{R}^p \to \mathbb{R}^q$ $\quad f(x) \in \mathbb{R}^q$, when $x \in \mathbb{R}^p$. $\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{x_p} \\ \vdots & & & \\ \frac{\partial f_q}{\partial x_1} & \cdots & & \frac{\partial f_q}{\partial x_p} \end{pmatrix}$

If $\varphi: \mathbb{R}^n \to \mathbb{R}$. is scalar-valued function.

Define $\nabla\varphi = [\frac{\partial\varphi}{\partial x}]^T \in \mathbb{R}^n$

Reading: Implicit Function Theorem

Skew Symmetric Matrices.

$so(n) = \{S \in \mathbb{R}^{n\times n} : S = -S^T\}$

If $M \in \mathbb{R}^{n\times n}$, we call $S = \frac{1}{2}(M - M^T)$ the "skew-symmetric part"

Note: $S^T = \frac{1}{2}[M - M^T]^T = \frac{1}{2}(M^T - M) = -\frac{1}{2}(M - M^T) = -S$

Likewise the symmetric part

$$Q = \frac{1}{2}(M + M^T)$$

we say M is "decomposed" into symmetric and skew-symmetric parts.

Hence, $S + Q = M$

Let $\wedge: \mathbb{R}^3 - so(3)$

$\vee: so(3) \to \mathbb{R}^3$

and $\vee$ call "ve" as to matrix

$$\hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

$S^\vee$

Two facts proved:

$$R\hat{v}R^T = \widehat{Rv}$$

Matrix Exponentials.

$$a^b = e^{\ln a^b} = \exp(b \ln a)$$

Taloy Series.

$$e^{A\theta} = I + A\theta + \frac{1}{2!}A^2\theta^2 + \cdots$$

Also Matrix log: $\ln A = I -$ something

Matrix log is an inverse of exp, on the image of $R^{n\times n}$ under exp.

Matlab

```
>> A = rand(3,3)
>> eA = expm(A);
>> Atest = logm(eA)
>> A - Atest
```

When we restrict $A \in \mathbb{R}^{3\times 3}$ to be a skew-symmetric matrix.

$A = \hat{w}\theta$. $\|w\| = 1$

Then $R = e^A = e^{\hat{w}\theta} = I + \sin\theta\hat{w} + (1-\cos\theta)\hat{w}^2$ is always a rotation.

And $\exp: so(3) \to SO(3)$ is onto $SO(3)$

Image of $so(3)$ under exp is $SO(3)$

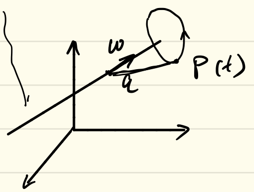
In this case, $\log(R)$

Inverse is not one-to-one.

Generalize exp to rigid Transformation via twists.

Motivation $\Rightarrow$ consider a pt $\in \mathbb{R}^3$ may be angular velocity by a unit vector w. $\|w\|=1$

$l = \{x \in \mathbb{R}^3 : x = q + \alpha w, \alpha \in \mathbb{R}\}$



$\dot{p} = w \times (p(t) - q)$

Let $\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0^T & 0 \end{bmatrix}$, $v = -w \times q$

$\Rightarrow \begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \hat{\xi} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{w} & v \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = w \times p + v = w \times (p - q)$

Let $\bar{p} = \begin{bmatrix} p \\ 1 \end{bmatrix}$ be homogeneous Coordinate for P.

$\dot{\bar{p}} = \hat{\xi} \bar{p}$ linear System fact.

$\Rightarrow \bar{p}(t) = \exp\{\hat{\xi} t\} \begin{bmatrix} p_0 \\ 1 \end{bmatrix}$

This motivates a generalization of $SO(3)$

$$se(3) = \{ (v, \hat{w}) : v \in \mathbb{R}^3, w \in \mathbb{R}^3 \}$$

3x3 skew symmetric.

$$se(3) = \left\{ \begin{bmatrix} \hat{w} & v \\ 0^T & 0 \end{bmatrix} : v \in \mathbb{R}^3, w \in \mathbb{R}^3 \right\}$$

Let $\xi = (v, w) \in \mathbb{R}^3$ be twist coordinates.

Let $\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0^T & 0 \end{bmatrix}$ and $\vee$ is the inverse.

overloaded hat

so given 4x4 twist S

$$S^\vee = (v, w) \in \mathbb{R}^6$$

Cool kid formula for $e^{\hat{\xi}\theta}$

$$e^{\hat{\xi}\theta} = I_{4\times4} + \theta\hat{\xi} + (1-\cos\theta)\hat{\xi}^2 + (\theta - \sin\theta)\hat{\xi}^3$$

Matrix Exponential of a twist:

$\xi = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^6 \qquad (\xi = (v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad , \quad \theta \in \mathbb{R}$

Assume $w \neq 0$, $\|w\| = 1$, without lost of generosity

?

$$\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0^T & 0 \end{bmatrix}$$

$$\exp\{\hat{\xi}\theta\} = I + \theta\hat{\xi} + (1-\cos\theta)\hat{\xi}^2 + (\theta - \sin\theta)\hat{\xi}^2$$

From MLS

Let $\hat{\xi} = (v, w) \in \mathbb{R}^3 \times \mathbb{R}^3$

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta + \frac{1}{2!}\hat{\xi}^2\theta^2 + \frac{1}{3!}\hat{\xi}^3\theta^3 + \cdots$$

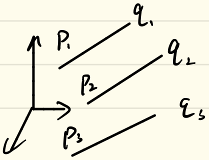
$$\hat{\xi}^n = \begin{bmatrix} \hat{w}^n & \hat{w}^{n-1} v \\ 0^T & 0 \end{bmatrix}$$

Case 1: Assume $w = 0$

$\hat{\xi}^n = 0$, except for $n = 1$, $\hat{\xi} = \begin{bmatrix} 0^{3\times3} & v \\ 0^T & 0 \end{bmatrix}$

$e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0^T & 1 \end{bmatrix}$ (can assume WOLOG $\|v\| = 1$

Translation of magnitude θ along v.



$q_i = e^{\hat{\xi}\theta} p_i$

$q_1 = p_1 + v\theta$

Case 2 : $\|w\| = 1$

Upper 3X3 Block is $e^{\hat{w}\theta} = I + \sin\theta\,\hat{w} + (1-\cos\theta)\hat{w}^2$

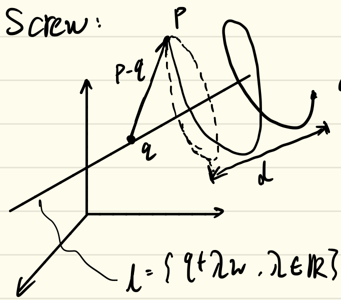
Derivation of upper 3X1 Block.

Let $g = \begin{bmatrix} R & d \\ 0^T & 1 \end{bmatrix} = \exp\{\hat{\xi}\theta\}$

$R =$ Rodrigues formula

$d = [I\theta + (1-\cos\theta)\hat{w} + (\theta - \sin\theta)\hat{w}^2]v$

Screw:



q : (any point on the axis)

$q + e^{\hat{w}\theta}(p-q) + h\theta w$

$h = \frac{d}{\theta}$ pitch of screw

A screw is given by $\{l, h, M\}$

axis pitch Magnitude (θ)

If $h = \infty$, then we have a pure translation.

$g = \begin{bmatrix} I & wM \\ 0^T & 1 \end{bmatrix}$

If $h \neq \infty$.

$g = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})q + h\theta w \\ 0^T & 1 \end{bmatrix}$

$$gp = \begin{bmatrix} e^{\hat{w}\theta} & (I-e^{\hat{w}\theta})q + h\theta w \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$= e^{\hat{w}\theta} p + (I - e^{\hat{w}\theta}) q + h\theta w$$

$$= e^{\hat{w}\theta}(p-q) + q + h\theta w$$

Let $q_2 = p + \lambda w$

Calculate g_2 based on q_2 (instead of q) and show $g = g_2$.

Given screw, compute twist.

Given a screw (l, h, θ), $h \neq 0$, $h \neq \infty$

$$\hat{\xi} = \begin{bmatrix} -w \times q + hw \\ w \end{bmatrix}$$

Recap:

screw: $\{ l, h, M \}$ — $l = \{ q + \lambda w, \lambda \in \mathbb{R} \}$, $\|w\| = 1$

- $h = \infty \Rightarrow$ pure translation

$$\Rightarrow g = \begin{bmatrix} I & wM \\ 0^T & 1 \end{bmatrix}$$

- $h \neq \infty$, $M = \theta$

$$\Rightarrow g = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})q + h\theta w \\ 0^T & 1 \end{bmatrix} \quad ☆$$

which looks similar to

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) + ww^T v\theta \\ 0^T & 1 \end{bmatrix} \quad *$$

$\xi = \begin{bmatrix} v \\ w \end{bmatrix}$, $\|w\| = 1$

To bring these into agreement, set $v = -(w \times q + hw)$

Plug $v \Rightarrow (*)$

$$(I - e^{\hat{w}t})[w \times (-w \times q + hw)] + ww^T[-w \times q + hw]\theta$$

$$= (I - e^{\hat{w}t})(-w \times (w \times q) + \cancel{hw \times w}^{\,0}) - (ww^T(\cancel{-w \times q})^{0} + (\underbrace{ww^Tw}_{1})h)\theta$$

$$= (I - e^{\hat{w}t})(-w \times (w \times q) + wh\theta)$$

$$= -(I - e^{\hat{w}t})\hat{w}^2 q + wh\theta$$

$$☆ = -(I - I - \hat{w}t - \frac{1}{2!}\hat{w}^2t^2 - \frac{1}{3!}\hat{w}^3t^3 - \cdots)\hat{w}^2 q$$

$$= \underbrace{+\hat{w}^2(\hat{w}t + \frac{1}{2!}\hat{w}^2\theta^2 + \cdots)q}$$

$\leftarrow$ n^{th} term: $\frac{1}{n!}\hat{w}^{n+2}t^n q$

$$\underbrace{w\times(w\times(w\times}_{n+2} q))$$

$$(-\underbrace{w\times(w\times(}_{n} w\times q))$$

make argument when computing Rodri.. Formula

Fold that back into expansion.

$$\star = -(\hat{w}t + \frac{1}{2}\hat{w}^2 t^2 + \dots) q$$

$$= +(I - e^{\hat{w}t}) q$$

Recap: finite pitch screw.

$\{l, h, M\} \longleftrightarrow \xi = \begin{bmatrix} v \\ w \end{bmatrix}$ with $v = -w \times q + hw$

see back for all cases. screw <=> twists.

(only one more case, $h = 0$)

Special Case

1) Zero pitch screw $\Rightarrow$ pure rotation about l

2) Infinite Pitch Screw $\Rightarrow$ pure translation

Usually, we factor twists like this: $g = e^{\hat{\xi}\theta}$

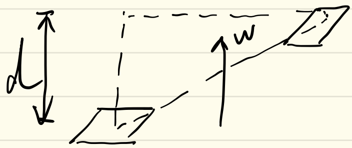
unit twist

i) $\|w\| = 1$

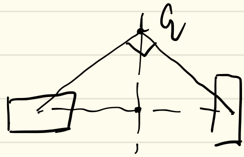
ii) $\begin{cases} \|w\| = 0 \\ \|v\| = 1 \end{cases}$

Charles Theorem

Every rigid body motion can be realized by a rotation about an axis, combined with a translation along that axis.



2D :



$h\frac{\pi}{2} = d \Rightarrow h = \frac{2d}{\pi}$ (pitch)

Proof of Charles Theorem, follows this logic:

1) If $g \in SE(3)$, $\exists \hat{\xi} \in se(3)$, st $e^{\hat{\xi}\theta} = g$

(From surjectivity of exp)

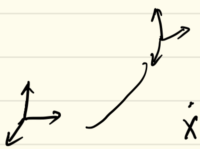
2) given $\hat{\xi}$ (twist), there is a corresponding screw motion

(prop. 2.10 in MLS)

Preview of rigid body velocities,

$R(t) = [X_b^a \; Y_b^a \; Z_b^a]$

Start w/ Rotation



$R(t) = SO(3)$, what is $\dot{R}$

$\dot{X}_b^a = w \times X_b^a$

$(R^T R) = I$

X_b^a : rep of x-axis of rotated form WRT original frame.

Rigid Body Velocity

start with rotational (angular) velocity.

Start with $R^TR = I$, where we assume $R(t) \in SO(3)$ is a smooth motion, ie. each element of $R(t)$ is differentiable in wrt t.

$$R(t) = \begin{bmatrix} r_{11}(t) & \cdots & \vdots \\ \vdots & & \vdots \\ \vdots & \cdots & r_{33}(t) \end{bmatrix} \Rightarrow \frac{d}{dt}R(t) = \dot{R}(t) \quad \text{and } ij \text{ th element is } \dot{r}_{ij}(t)$$

$\dot{R}$ by itself is not a great way to keep track of rotational velocities.

starting with $(*)$ $R^TR = I$

$$\downarrow$$

$$\frac{d}{dt}R^TR = \cancel{\frac{d}{dt}I} \rightarrow 0$$

side fact: if $A(t)$, $B(t)$ have compatible dimensions, and are smooth functions of time,

$$\frac{d}{dt}(AB) = \dot{A}B + A\dot{B}$$

$$\Rightarrow \frac{d}{dt}(R^TR) = \frac{d}{dt}R^TR + R^T\frac{d}{dt}R = 0_{3\times3}$$

$$= \dot{R}^TR + R^T\dot{R}$$

$$= (R^T\dot{R})^T + R^T\dot{R}$$

$$\Rightarrow (R^T\dot{R})^T = -R^T\dot{R}$$

$\Rightarrow$ $R^T\dot{R}$ is a skew-symmetric matrix.

Therefore, we can define $\omega^b = (R^T\dot{R})^\vee$ as so called Body Angular Velocity

Likewise, starting with $RR^T = I$

$$\downarrow$$

$$\frac{d}{dt} \cdots$$

$$\downarrow$$

$$\omega^s = (\dot{R}R^T)^\vee \quad : \quad \text{Spacial Angular Velocity}$$

Look at $\hat{\omega}_b = R^T\dot{R}$

Premultiply by R . post-multiply by R^T on both sides.

$$R\hat{\omega}_b R^T = \underbrace{R(R^T}_{I}\underbrace{\dot{R})R^T}_{\hat{\omega}^s} = \hat{\omega}^s$$

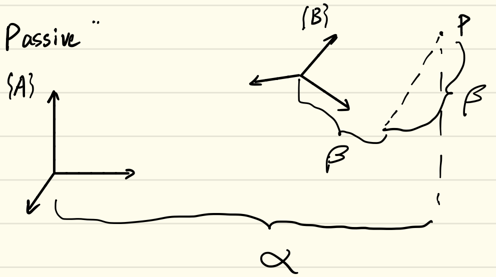
$$\Rightarrow \hat{\omega}^s = R(\hat{\omega}^b)R^T$$

$$\Rightarrow \omega^s = R\omega^b$$

Likewise, $\omega^b = R^T\omega^s$

Two interpretations of rigid transformation.

· Passive ··



P_b = rep of P wrt $\{B\}$

P_a = ... $\{A\}$

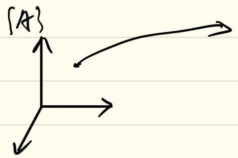
$P_a = \begin{bmatrix}\alpha \\ 0\end{bmatrix} \qquad P_b = \begin{bmatrix}\beta \\ \beta\end{bmatrix}$

Passive intepretation of $g_{a,b} \in SE(3)$ is the change of coordinate.

$$P_a = g_{ab} P_b$$

Active intepretation of $g_{ab} \in SE(3)$ is the movement of the points via mult by g_{ab}.

$$P \to gP$$

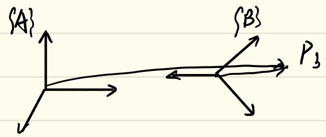


Consider motion of a point under a motion $P_{ab}(t)$

$$P_a(t) = R_{ab}(t) \underbrace{P_b}_{\text{constant.}}$$

velocity of point

$$V_a = \dot{P}_a = \dot{R}_{ab} P_b + R_{ab} \cancelto{0}{\dot{P}_b}$$



$$\Rightarrow \quad V_a = \dot{R}_{ab} P_b$$

Body velocity

$$V_b \neq \frac{d}{dt} P_b = 0$$

Rather: $V_b = \hat{w}^b_{ab} P_b = w^b_{ab} \times P_b$

Put v & P in same frame

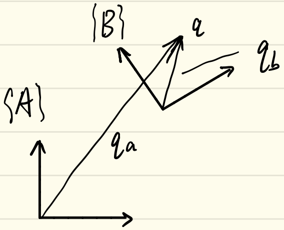
Note: $P_b = R^T_{ab} P_a$ $\quad \Rightarrow V_a = \dot{R}_{ab}(R^T_{ab} P_a) = (\dot{R}_{ab} R^T_{ab}) P_a$

$(P_a = R_{ab} P_b)$

$$V_a = \hat{w}^s_{ab} P_a = w^s_{ab} \times P_a$$

SE(3) Motion

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & P_{ab}(t) \\ 0^T & 1 \end{bmatrix}$$



$q_b \equiv$ const.

$$V_{q_a} = \frac{d}{dt} q_a(t) \quad \text{(B/C \{A\} is considered stationary)}$$

$$\Rightarrow V_{q_a} = \frac{d}{dt}(g_{ab}(t) q_b)$$

$$= \dot{g}_{ab} \underbrace{q_b}_{g_{ab}^{-1} q_a}$$

$$\Rightarrow V_{q_a} = \underbrace{(\dot{g}_{ab} g_{ab}^{-1})}_{\hat{V}^s_{ab}} q_a$$

$\hat{V}^s_{ab}$ = spatial Body Velocity.

Let $g(t) = \begin{bmatrix} R(t) & P(t) \\ 0 & 1 \end{bmatrix}$

Twist

$$\dot{g} g^{-1} = \begin{bmatrix} \dot{R} & \dot{P} \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T P \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T P + \dot{P} \\ 0^T & 0 \end{bmatrix}$$

$$\Rightarrow V^s = \begin{bmatrix} v^s \\ w^s \end{bmatrix} = \begin{bmatrix} -\dot{R}R^T P + \dot{P} \\ (\dot{R}R^T)^\vee \end{bmatrix}$$

ie. $\hat{V}^s = \dot{g}g^{-1}$

$$\Rightarrow V^s = (\dot{g}g^{-1})^\vee$$

and likewise,

$$\hat{V}^b = g^{-1}\dot{g}$$

$$\Rightarrow V^b = (g^{-1}\dot{g})^\vee$$

where $\hat{V}^b = \begin{bmatrix} R^T & -R^T p \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0^T & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0^T & 0 \end{bmatrix}$

$$V^b = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix} = \begin{bmatrix} v^b \\ w^b \end{bmatrix}$$

Go back to $V_{q_a} = \underbrace{(\dot{g}_{ab}\, g_{ab}^{-1})}_{\hat{V}^s_{ab}} q_a$

$$= \begin{bmatrix} \hat{w}^s_{ab} & v^s_{ab} \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} q_a \\ 1 \end{bmatrix}$$

$$= \hat{w}^s_{ab}\, q_a + v^s_{ab}$$

where $\hat{w}^s_{ab} = \dot{R}_{ab} R^T_{ab}$

$$v^s_{ab} = -\dot{R}_{ab} R^T_{ab} P_{ab} + \dot{P}_{ab}$$

Recap.

$$\hat{V}^s = \dot{g} g^{-1} \Rightarrow V^s = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T)^\vee \end{bmatrix}$$

$$\hat{V}^b = g^{-1}\dot{g} \Rightarrow V^b = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix}$$

These are related by a similarity transformation!

$$g^{-1}\hat{V}^s g = \underbrace{g^{-1}(\dot{g}}_{\hat{V}^b}\underbrace{g^{-1})g}_{I} = \hat{V}^b$$

$$\hat{V}^b = g^{-1}\hat{V}^s g$$

$$\hat{V}^s = g\hat{V}^b g^{-1}$$

Can we turn this into a relationship between V^s V^b

See P55 - P56 MLS

$\Downarrow$

$$V^s = \underbrace{\begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}}_{6\times 6 \text{ matrix}} V^b \Rightarrow V^b = (Ad_g)^{-1} V^s$$

$$\begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} = Ad_g$$

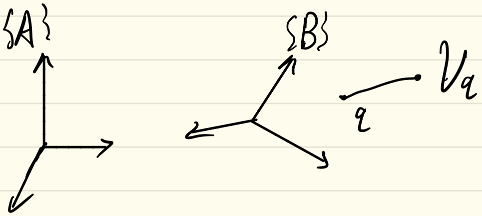
$$= \underbrace{\begin{bmatrix} R^T & -(\widehat{R^T p})R^T \\ 0 & R^T \end{bmatrix}}_{Ad_{g^{-1}}} V^s$$

$$\Rightarrow Ad_{g^{-1}} = [Ad_g]^{-1}$$

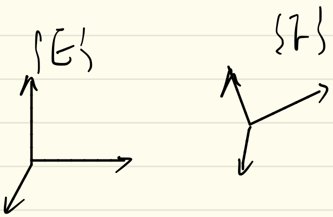
"Inverse of Adjoint is the Adjoint of the inverse"

Recall $v_{q_a} = \hat{V}^s_{ab} q_a$

What is vel of q written wrt $\{B\}$? (Instantaneous Frame)



$$v_{q_b} = (g_{ab})^{-1} v_{q_a}$$



$$v_{q_e} = \hat{V}^s_{ef} q_e$$

$$v_{q_f} = (g_{ef})^{-1} v_{q_e}$$

$$v_{q_b} = g_{ab}^{-1} \hat{V}^s_{ab} q_a$$

$$= \underbrace{g_{ab}^{-1} \hat{V}^s_{ab} g_{ab}}_{\hat{V}^b_{ab}} q_b$$

$$\Rightarrow v_{q_b} = \hat{V}_{ab} q_b$$

Velocity of a screw motion

Let $g(t) = e^{\hat{\xi}\theta(t)} g(0)$ $\qquad g(0) = \text{const}$

$\Rightarrow \dot{g} = \hat{\xi}\dot{\theta} e^{\hat{\xi}\theta} g(0)$ $\qquad \Rightarrow \hat{V}^s = \dot{g} g^{-1} = \hat{\xi}\dot{\theta} e^{\hat{\xi}\theta} g(0) \left[e^{\hat{\xi} t} g(0) \right]^{-1}$

$= \hat{\xi}\dot{\theta}$

However, $\hat{V}^b = \dot{g} g^{-1} = g_1^{-1} \hat{\xi} g(0) \dot{\theta} \Rightarrow$ $\qquad V^b = (Ad_{g_1^{-1}} \xi)\dot{\theta}$

Does the choice of Body Frame affect V^b ?

$$\hat{V}_{ac}^b = g_{ac}^{-1} \cdot \dot{g}_{ac}$$

$$= (g_{bc}^{-1} \underbrace{g_{ab}^{-1}) (\dot{g}_{ab}}_{\hat{V}_{ab}^b} g_{bc})$$

$$g_{bc}^{-1} \quad \hat{V}_{ab}^b \quad g_{bc}$$

$$\therefore V_{ac}^b = Ad_{g_{bc}^{-1}} V_{ab}^b$$

A: Yes, Body Velocity is not Invariant to body Frame.

Spacial Velocity.

$$\hat{V}_{ac}^s = \dot{g}_{ac} g_{ac}^{-1} = (\dot{g}_{ab} g_{bc})(g_{ab} g_{bc})^{-1} = \dot{g}_{ab} \overbrace{g_{bc} g_{bc}^{-1}}^{I} g_{ab}^{-1} = \hat{V}_{ab}^s$$

$\Rightarrow$ Invariant

Sum up:

spacial velocity: • is invariant to body frame
• is NOT invariant to spacial frame.

{A}, {B} Fixed {C} Move

Calculate $\hat{V}^s_{ac} = \dot{g}_{ac} g_{ac}^{-1} = (\cancel{\dot{g}_{ab}}^{0} g_{bc} + g_{ab}\dot{g}_{bc})(g_{bc}^{-1} g_{ab}^{-1})$

$$= g_{ab} \underbrace{\dot{g}_{bc} g_{bc}^{-1}}_{} g_{ab}^{-1}$$

$$g_{ab} \hat{V}^s_{bc} g_{ab}^{-1}$$

$$\Rightarrow V^s_{ac} = Ad_{g_{ab}} V^s_{bc}$$

$\downarrow$

Not invariant to spacial frame.

Body Velocity: • is invariant to spacial frame $\Rightarrow V^b_{ac} = V^b_{bc}$
• is not invariant to body frame, $\Rightarrow V^b_{ac} = Ad_{g_{bc}^{-1}} V^b_{ab}$

Suppose {A} is fixed, {B} moves relative {A} $\Rightarrow g_{ab}(t)$
{C} moves relative {B} $\Rightarrow g_{bc}(t)$

$\Rightarrow$ {C} ... {A} $\Rightarrow g_{ac} = g_{ab} \cdot g_{bc}$

$$\hat{V}^s_{ac} = \dot{g}_{ac} g_{ac}^{-1} = (\dot{g}_{ab} g_{bc} + g_{ab}\dot{g}_{bc})(g_{bc}^{-1} g_{ab}^{-1})$$

$$= \dot{g}_{ab} \cancel{g_{bc} g_{bc}^{-1}} g_{ab}^{-1} + g_{ab} \dot{g}_{bc} g_{bc}^{-1} g_{ab}^{-1}$$

$$\hat{V}^s_{ac} = \hat{V}^s_{ab} + g_{ab} \hat{V}^s_{bc} g_{ab}^{-1}$$

$$\Rightarrow \quad V_{ac}^{s} = V_{ab}^{s} + Ad_{g_{ab}} V_{bc}^{s}$$

Similarly, $V_{ac}^{b} = V_{ac}^{b} + Ad_{g_{bc}^{-1}} V_{ab}^{b}$

Adjoint.

- $V^{s} = Ad_{g} V^{b}$
- $V_{ac}^{s} = V_{ab}^{s} + Ad_{g_{ab}} V_{bc}^{s}$
- $V_{ac}^{b} = V_{bc}^{b} + Ad_{(g_{bc}^{-1})} V_{ab}^{b}$
- $\xi' = Ad_{g} \xi$ (Transform of screw)

Overview :

- Points . vectors
- Rigid Transformations
- Groups : group action. group product. homomorphism.
- Representing rigid transformation
  - Homog Transformations
  - Exponentials of Twists (Rodrigues, cool-kid formula)
  - Euler Angles , $e^{\hat{w}_x \theta}$. $w_x = \left|\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right|$
  - Axis . Angle
  - screw $\Leftrightarrow$ twists
- Rigid Body velocities.
- Adjoints.
- DOF : constraints.

- Linear Vector Spaces - linearity.

$G: SO(2) \to SO(3)$

$$G(R) = \begin{bmatrix} R & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \; 0 & 1 \end{bmatrix}$$

- Twists as velocites.

Final 12/21

This class : serial - link, open chain manipulators, typically with a sequence of single DOF joints

Joints can be

- 1 DOF
  - revolute
  - Prismatic
  - Helical

  } Screw Motion

- 2DOF

  cylindrical
  rev. + Prismatic

- 3 DOF
  - Planar
  - Spherical

  ball in a socket.

"Lower Pairs"

Goal in Forward Kinematics: Find a mapping from joint space to configuration space

$$g: Q \rightarrow SE(3)$$

where Q = joint space. $Q = \mathbb{R}^n$, where n = # of joints.

(Technically, $Q = T^n$ for all revolute joint: $T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$

We say if $Q = \mathbb{R}^n$, then it's an n-DOF robot.

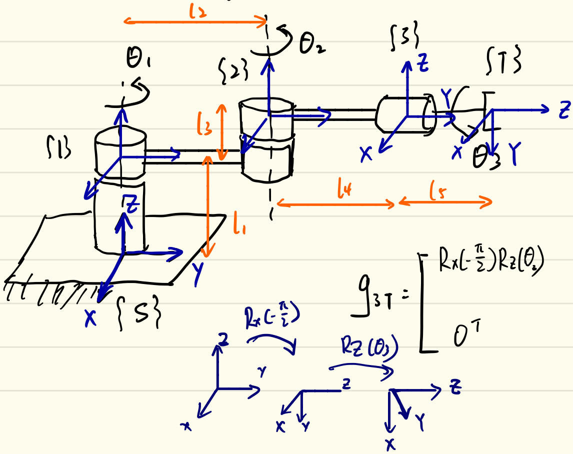
If $n < 6$, under actuated robot

$n = 6$ Fully actuated robot.

$n > 6$, redundant manipulator

Classical Kinematics: eg. in Craig

Attach Frames to each Link. calculate (passive) link-to-link transformation.



$$g_{3T} = \begin{bmatrix} R_x(-\frac{\pi}{2})R_z(\theta_3) & \begin{matrix} 0 \\ l_5 \\ 0 \end{matrix} \\ 0^T & 1 \end{bmatrix}$$

$$g_{s1}(\theta_1) = \begin{bmatrix} R_z(\theta_1) & \begin{matrix} 0 \\ 0 \\ l_1 \end{matrix} \\ 0^T & 1 \end{bmatrix}$$

$$g_{12}(\theta_2) = \begin{bmatrix} R_z(\theta_2) & \begin{matrix} 0 \\ l_2 \\ l_3 \end{matrix} \\ 0^T & 1 \end{bmatrix}$$

$$g_{23}(\theta_3) = \begin{bmatrix} I & \begin{matrix} 0 \\ l_4 \\ 0 \end{matrix} \\ 0^T & 1 \end{bmatrix}$$

(D-H came up a standard procedure.)

Final Mapping

$$g_{st}(\theta_1, \theta_2, \theta_3) = g_{s1} \cdot g_{12} \cdot g_{23} \cdot g_{3T}$$

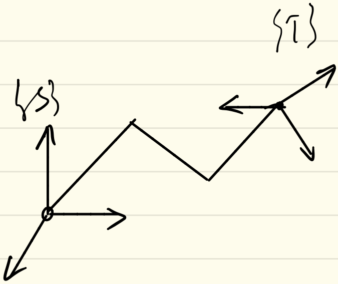
Transform points from tool frame $\underset{g_{st}^{-1}}{\overset{g_{st}}{\rightleftarrows}}$ spatial frame

Active Persepective.

Product of Exponential Formulation.

Define two frames.

①



② Choose a home configuration, and call this $\theta = 0. \in \mathbb{R}^n$

③ Calalate ξ

④ POE

$$g_{st}(\theta) = e^{\hat{\xi}\theta_1} \cdots e^{\hat{\xi}\theta_n}$$

Correction : $f: Q \longrightarrow SE(3)$

Q: C-space or joint space

$SE(3)$: contains workspace.

Product of Exponential Formula.

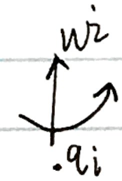
$$g(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

STEP 1 : Place $\{S\}$ and $\{t\}$ to tool

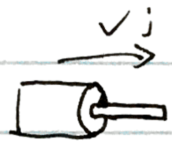
STEP 2 : Pick $\theta = 0$ configuration. and compute $g_{st}(0)$

STEP 3 : At $\theta = 0$, determine $\xi_1 \cdots \xi_n$.
(e.g. by calculating the associated screws) w.r.t. $\{S\}$

STEP 4 : $g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)$



$$\xi = \begin{bmatrix} -w_i \times q_i \\ w_i \end{bmatrix}$$



$$\xi = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$$

Does the order matter ?

$$e^{\hat{\xi}_n\theta_n} \cdots e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_1\theta_1} g_{st}(0) \stackrel{?}{=} g_{st}(\theta)$$

Look @ $n=2$ case

After applying a rotation θ_1 the axis associated with ξ_1

How do we move $\xi_2 \to \xi_2'$

$$\xi_2' = Ad_{e^{\hat{\xi}_1\theta_1}} \xi_2$$

$$\Rightarrow e^{\hat{\xi}_2'\theta_2} = e^{(Ad_{e^{\hat{\xi}_1\theta_1}} \xi_2)^{\wedge}\theta_2}$$

$$= \exp\{ e^{\hat{\xi}_1\theta_1} \hat{\xi}_2 [e^{\hat{\xi}_1\theta_1}]^{-1} \theta_2 \}$$

$$\boxed{\begin{aligned} &\text{Aside: } e^{ABA^{-1}} = I + ABA^{-1} + \frac{[ABA^{-1}]^2}{2!} + \cdots \\ &(ABA^{-1})^n = AB^nA^{-1} \\ &= A(e^B)A^{-1} \end{aligned}}$$

$$= e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$\Rightarrow$ For $n=2$, $e^{\hat{\xi}_2'\theta_2} e^{\hat{\xi}_1\theta_1} g_{st}(0)$

$$= [e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}] e^{\hat{\xi}_1\theta_1} g_{st}(0)$$

$$= e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} g_{st}(0)$$

Jacobian: Mapping from joint-space velocities to end-effector velocities. It's calculated using chain rule.

"Easy" Case: End-effector pose is written using local condition, eg. position + Euler angle

$$r = \begin{bmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{position} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Euler angles.} \end{matrix}$$

$$r = f(q), \quad f: \underset{\text{joint-space}}{\mathbb{R}^6} \rightarrow \underset{\text{pose}}{\mathbb{R}^6}$$

$$\dot{r} = \overset{J(q)}{\boxed{\frac{\partial f}{\partial q}}} \cdot \dot{q}$$

Using Euler angles introduces artificial singularities!

To avoid this, we compute the manipulator Jacobian based on either the body velocity V_{st}^{b} or the spatial velocity V_{st}^{s}

$$V_{st}^{b} = J_{st}^{b}(q)\,\dot{q}$$

$$V_{st}^{s} = J_{st}^{s}(q)\,\dot{q}$$

The Jacobian is a linear Mapping between joint-space velocities and end-effector velocities

To calculate V_{st}^b or V_{st}^s, go back to the definition.

$$\hat{V}_{st}^b = g_{st}^{-1}\dot{g}_{st}$$

$$\hat{V}_{st}^s = \dot{g}_{st}\, g_{st}^{-1}$$

POE

$$g_{st}(q) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

Start with

$$\hat{V}_{st}^s = \frac{d}{dt}\left[e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)\right]\left[e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)\right]^{-1}$$

$$= \hat{\xi}_1\dot{\theta}_1 e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) + e^{\hat{\xi}_1\theta_1}\hat{\xi}_2\dot{\theta}_2 e^{\hat{\xi}_2\theta_2} \cdots g_{st}(0)$$

$$+ \cdots + e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_{n-1}\theta_{n-1}} \dot{\theta}_n \hat{\xi}_n e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

Multiply together

$$= \dot{\theta}_1\hat{\xi}_1 + \dot{\theta}_2 e^{\hat{\xi}_1\theta_1}\hat{\xi}_2 e^{-\hat{\xi}_1\theta_1} + \dot{\theta}_3 e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2}\hat{\xi}_3 e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1}$$

$$+ \cdots + \dot{\theta}_n e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{n-1}\theta_{n-1}}\hat{\xi}_n e^{\hat{\xi}_{n-1}\theta_{n-1}} \cdots e^{\hat{\xi}_1\theta_1}$$

linear in $\dot{\theta}_i$

$$\xi_i' = Ad_{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}} \hat{\xi}_i$$

$$\left(q = \begin{bmatrix}\theta_1 \\ \vdots \\ \theta_n\end{bmatrix}\right)$$

"Vee" $\Rightarrow V_{st}^s = \dot{\theta}_1\xi_1 + \dot{\theta}_2\xi_2' + \cdots + \dot{\theta}_n\xi_n'$,

$$\Rightarrow \underset{\in \mathbb{R}^6}{V_{st}^s} = \underbrace{\left[\xi_1 \cdots \xi_2' \cdots \xi_n' \right]}_{J_{st}^s(q)} \underbrace{\begin{bmatrix} \dot\theta_1 \\ \vdots \\ \dot\theta_2 \\ \vdots \\ \dot\theta_n \end{bmatrix}}_{\dot q \in \mathbb{R}^n}$$

see back for $J_{st}^b(q)$

Recall

$$\hat{V}_{st}^b = g_{st}^{-1}\dot g_{st} = g_{st}^{-1}\overbrace{\left[\dot g_{st}\, g_{st}^{-1}\right]}^{I} g_{st} = \overbrace{Ad_{g_{st}^{-1}} V_{st}^s}$$

$$\Rightarrow J_{st}^b(q) = Ad_{g_{st}^{-1}} J_{st}^s$$

Recall If $g_{st}(q)$ is the forward kinematic map and say $J_{st}^s(q)$ is the spatial Jacobian matrix.

A singularity is a point $q_{st} \in Q$, st $J(q)$ loses rank from its usual rank.

If $Q \in \mathbb{R}^6$, then look at $\det(J(q))$ and if $\det(J(q))$ is a non trivial (scalar) function, then the usual rank is 6.

The job of finding singularities reduces to finding q vector that cause $\det(J(q)) = 0$

Other cases:

$n < 6$: $\text{Rank}(J) \leq n < 6$

Assume rank(J) is "usually" n, then a singularity would be when $\text{rank}(J) < n$

$n > 6$: redundant manipulator,

Typically $\text{rank}(J) = 6$

- Enhance dexterous workspace.
- provides extra to solve sub-tasks. (say, avoiding collision)
- singularity when rank < 6

Two physical Ramifications of singularities.

1) Loss of (local) DOF of end-effector.

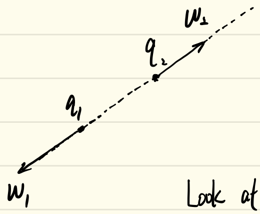
(can't achieve certain end-effector velocities)

2) Inability to transmit certain end-effector forces into torques

Characterizing singularities using twists.

Recall for all revolute robots, $J(q) = \begin{bmatrix} -w_1 \times q_1 & -w_2 \times q_2 & \cdots & -w_n \times q_n \\ w_1 & w_2 & & w_n \end{bmatrix} \in \mathbb{R}^{6 \times n}$

Example 1 co-linear parrallel joints.



① $w_1 = \pm w_2$

② $w_i \times (q_1 - q_2) = 0$

say $w_1 = w_2$

Look at $\xi_1 - \xi_2 = \begin{bmatrix} -w_1 \times q_1 \\ w_1 \end{bmatrix} - \begin{bmatrix} -w_2 \times q_2 \\ w_2 \end{bmatrix}$

$$= \begin{bmatrix} -w_1 \times (q_1 - q_2) \\ 0 \end{bmatrix}$$

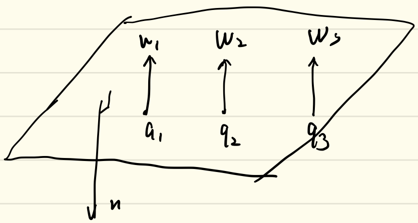
if $w_1 = -w_2$, then add $\xi_1 + \xi_2$

. ξ_1 and ξ_2 are linealy dependent.

Example 2: parallel co-planar revolute joints.

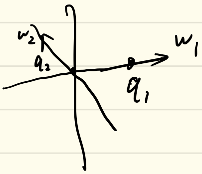
$\Rightarrow$ 1) $w_i = \pm w_j$, $i, j = 1, 2, 3$

2) $\exists\, n \neq 0$. $n^T (q_1 - q_2) = 0$



Example 3. 4 co-intersecting revolute

$w_i \times (q_i - q) = 0$ for some q.



Example 4. (Not in book)

Four Parallel (not nec. co-planar) axes.

for some w.

Assume $w_i = w$. $i = 1 \cdots 4$.

If any are neg, deal w/ on sep time)

$$J = \begin{bmatrix} -w_1 \times q_1 & -w_2 \times q_2 & -w_3 \times q_3 & -w_4 \times q_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}$$

Follow through mathematica notebook.

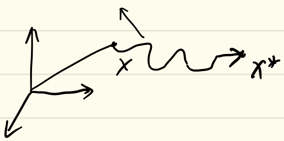
subtract last col from 1st three

$\hat{w}[a\,b\,c]$ for upper left 3x3 block.

What are Jacobian Matrices good for?

First: A rudimentary Cartesian controller. Let $x \in \mathbb{R}^3$ be the position of a point under our control.

$\dot{x} = u \quad u \in \mathbb{R}^3$, where u is our control input.



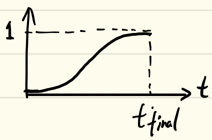
Suppose our goal is to drive $x(t) \to x^*$, where $x \in \mathbb{R}^3$ is a fixed goal.

"CS" approach.

Compute a path for our initial state to x^*. $x_p(t)$

$$x_p(t) = \alpha(x^*) + (1-\alpha)x$$

$\alpha(t)$



Now let $u(t) = \dfrac{d}{dt} x_p(t)$

"EE" approach : a feedback algorithm.

Define a control policy.

$$u(t) = K(x^* - x(t))$$

scalar feedback gain

$$[K] = \frac{m/s}{m} = \frac{1}{s}$$

$$\Rightarrow \dot{x} = u = K[x^* - x]$$

Error

$$e(t) = x - x^*$$

$$\Rightarrow \frac{d}{dt} e = \dot{e} = \dot{x} - \cancel{\dot{x}^*}^{\,0} = -ke = -k\,e$$

$$(\dot{x} = Ax \Rightarrow x(t) = e^{At} x_0$$

$$e(t) = e^{-kt} e_0$$

$$\Rightarrow e(t) = 0 \text{ as } t \to \infty$$

Pick K so that $\tau = \frac{1}{K}$ is short enough for our purpose.

Asymptotic convergence practicing at x^* w/in st.

Resolved rate control

Let $x = f(q)$, $\dot{q} = u$. where $q \in \mathbb{R}^3$ (joint variable)

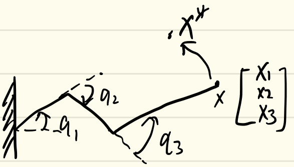
control model.

$x \in \mathbb{R}^3$ tool-tip position.

Define $J(q) = \frac{\partial f}{\partial q} \in \mathbb{R}^{2\times3}$

Recall $\dot{x} = \underbrace{J(q)}_{\text{linear map.}}\dot{q}$

$\Rightarrow \dot{q} = J^{-1}(q)\,\dot{x}$



Calculate algorithm:

$u = kJ^{-1}(q)[x^* - x]$

joint space velocities.

chain rule.

Plug in $\Rightarrow \dot{x} = \frac{d}{dt}f(q) = J(q)\cdot\dot{q}$

$\Rightarrow \dot{x} = J(q)u = J(q)[kJ^{-1}(q)\cdot[x^*-x]]$

$\Rightarrow \dot{x} = k[x^* - x]$

from the force.

$$X \to X^* \quad \text{as} \quad t \to \infty$$

Must worry about singularities

Discrete time implementation

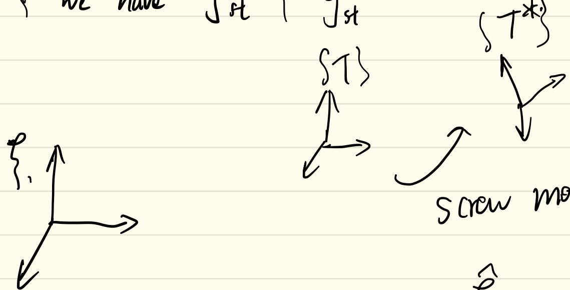
$$\frac{q(k+1) - q(k)}{\Delta T} \approx \dot{q} = K J^{-1}(q)[X^* - X]$$

$$\Rightarrow q(k+1) = q(k) + \underbrace{\Delta T\, K}_{\text{two variable.}} J^{-1}(q)[X^* - X]$$

6 DOF POE generalization:

$$g_{st}(q) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_6\theta_6} g_{st}(0)$$

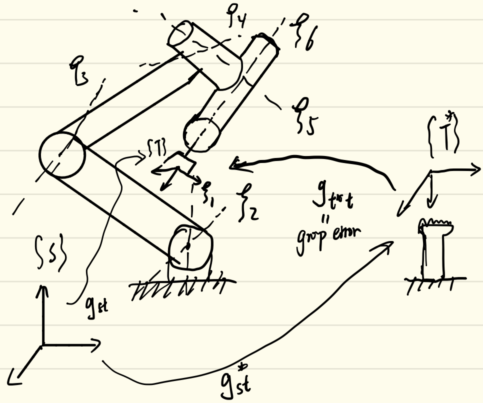
; we have J^b_{st} ; J^s_{st}



screw motion. from $g_{st} \to g^*_{st}$

$$g_{st}(g^*_{st})^{-1} = e^{\theta \hat{\xi}_e}$$

, where ξ_e is the twist coordinate vector. of the group error.



$$e = x - x^*$$

$$g_{st} = g_{st^*}\, g_{t^*t}$$

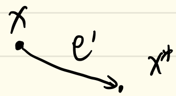
$$\underbrace{g_{t^*t}}_{\text{error}} = g_{st^*}^{-1}\, g_{st}$$

We want $g_{t^*t} \to I$

At time $t=0$, suppose $g_{t^*t} = e^{\hat{\xi}_0 \theta}$. where ξ_0, θ are extracted using the usual formula.

Our goal: Make g_{t^*t} stay along the same screw (associated with ξ_0)

but make $\theta(t) \to 0$ as $t \to \infty$



$$e' = x^* - x$$

$$\dot{e}' = \cancel{\dot{x}^*}^{\,0} - \dot{x} = -u$$

$$\Rightarrow u = -k(x - x^*) = k\overbrace{(x^* - x)}^{e'} = +ke'$$

$$\Rightarrow \dot{e}' = -u - (ke') = -ke'$$

Idea: make $V^b_{t^*t}$ point along the screw

$$\hat{V}^b_{t^*t} = g_{t^*t}^{-1}\, \dot{g}_{t^*t} = \underbrace{g_{st}^{-1}\, g^*_{st}}_{I}\, \underbrace{[\, g_{st^*}^{-1} \quad \dot{g}_{st}\,]}_{\dot{g}_{t^*t}}$$

$$= g_{st}^{-1} \dot{g}_{st} = \hat{V}_{st}^{b}$$

$$V_{t^*t}^{b} = V_{st}^{b} = J_{st}^{b}(q)\,\dot{q}$$

We will use this later to determine $\dot{q}$. First find a good choice for V^b.

Let $V_{t^*t}^{b} = -k\xi$, where ξ is the current unnormalized twist, ie. $\xi = \xi_0\theta$

We will now show $\dot{\xi} = -k\xi$

If $\dot{\xi} = -k\xi$

$$\Rightarrow \xi(t) = e^{-kt}\underbrace{\xi_0\theta_0}_{\text{initial condition.}}$$

$$\Rightarrow g_{t^*t} = \exp(\underbrace{\theta_0 e^{-kt}}_{\theta(t)}\hat{\xi}_0)$$

$$\Rightarrow \underbrace{g_{t^*t}^{-1}\dot{g}_{t^*t}}_{V^b=} = e^{-\theta(t)\hat{\xi}_0}[\dot{\theta}\hat{\xi}e^{\theta(t)\hat{\xi}_0}]$$

$$= \dot{\theta}\hat{\xi}$$

$$\Rightarrow V^b = \dot{\theta}\xi = -k\theta_0 e^{-kt}\xi$$

$$= -k\xi$$

aside:

could choose

$$V_{t^*t} = \frac{k\xi}{\|\xi\|+\varepsilon}$$

$\|\xi\|$ larger than ε

$$V_{t^*t} = -k\frac{\xi}{\|\xi\|}$$

$\Rightarrow$ const velocity

when $\|\xi\| < \varepsilon$

$$\Rightarrow V_{t^*t} = -\frac{k}{\varepsilon}\xi$$

$\Rightarrow$ if we choose $V^b_{st} = V^b_{t^*t} = -k\xi$

measure at every time step.

$\Rightarrow \dot{\xi} = -k\xi(t)$

$\Rightarrow \xi \to 0$ as $t \to \infty$

$\Rightarrow g_{t^*t} \to I$

$\Rightarrow \dot{q} = J^b_{st}(q)^{-1}[-k\xi]$ Resolved rate control formula

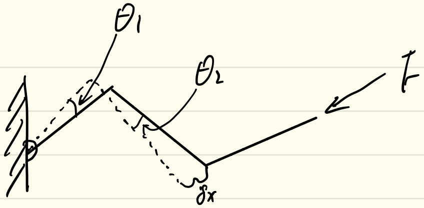
MIT Linear Alg G.S.

The transpose Jacobian

Generalized Forces $\tau \in \mathbb{R}^n$ ($n = 4$ joints) are just quantity s.t. the virtual work δW done by virtual displacement $\delta\theta$ is given by

$$\delta W = \sum_{i=1}^{n} \tau_i \delta\theta_i$$

At the end-effector, the virtual work is given by $\delta W = F \cdot \delta x$, where δx is a virtual displacement.



Conservation of energy $\Rightarrow$ Principal of Virtual Work:

We can equate virtual work at end-effector with the virtual work done by joints.

$$\delta W = F \cdot \delta x = \tau \cdot \delta \theta$$

vector dot product

$$= \underbrace{F^T}_{\in R^3} \underbrace{\delta x}_{\in R^3} = \underbrace{\tau^T}_{\in R^n} \underbrace{\delta\theta}_{\in R^n}$$

Let's assume $x = f(\theta)$, $f: R^n \to R^3$. forward kinematics.

$$\Rightarrow \delta x = \underbrace{\left| \frac{\partial f(\theta)}{\partial \theta} \right|}_{J(\theta)} \delta \theta$$

Infinite decimal

$$\Rightarrow \underbrace{\overbrace{F^T}^{1\times 3} \overbrace{J(\theta)}^{3\times n}}_{1\times n} \delta\theta = \underbrace{\tau^T}_{1\times n} \delta\theta$$

Can not mindlessly cancel $\delta\theta$.

but since the expression is true for all $\delta\theta$ $\Rightarrow$

$$F^T J(\theta) = \tau^T$$

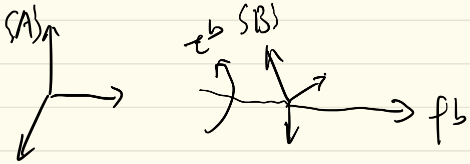
$$\Rightarrow \tau = [J(\theta)]^T F$$

To generalize this to Force / $\underbrace{\text{Torque}}_{\text{moment}}$ @ end-effector.

We need a way of expressing

Force/torques: WRENCHES.

$$F^b = \begin{bmatrix} f^b \\ \tau^b \end{bmatrix} \quad \begin{matrix} \leftarrow \text{linear forces.} \\ \leftarrow \text{Torque.} \end{matrix}$$



Body Wrench can be written wrt other body frames.

We can also define a spatial frame. See Chapter 2

F^b is define so that $\delta W = F^b \cdot \underbrace{V^b \delta t}_{\substack{\text{virtual displacement} \\ \text{(not a velocity)}}}$ ← Noah's fix on unit.

Now, for a n-DOF robot with Kinematics

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)$$

We have $J^b_{st}(\theta)$, s.t. $\underbrace{V^b \delta t}_{\text{virtual displacement}} = J^b_{st}(\theta)\delta\theta$

$$\delta W = (F^b)^T J_{st}^b(\theta)\,\delta\theta$$

and, like before

$$\delta W = \tau^T \delta\theta$$

$$\Rightarrow \underbrace{\tau}_{\text{joint torque}} = J_{st}^b(\theta)^T F^b$$

$g = \ldots$,

$\underbrace{x_i}_{6\times1} = g_0 + X_i(g)$

$g_{test} = \exp(\mathrm{hat}(X_i))$,

$(g_{test} \backslash g)$ ml divide

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$AB = C$

$B = A^{-1}C$

$B = A \backslash C$

$B = \mathrm{inv}(A) * C$

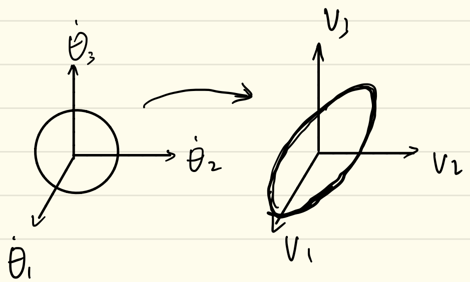
Manipulability:

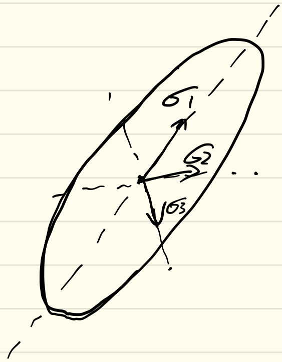
Affordance over the velocities of end-effector.

$\uparrow$ Manipulability $\Rightarrow$ can achieve 6 DOF of velocity of end-effector with reasonable joint velocities.

Recall that the Jacobian is (of any joint configuration) linear mapping

$$V = J(\theta)\dot{\theta}$$





σ_i's are the square roots of the non-zero eignvalues of

$(J^T J)$ or $(J J^T)$

look at case $n=6$, $J \in \mathbb{R}^{6\times6}$

Look at $\|V\|^2 = V^T V = (J\dot{\theta})^T (J\dot{\theta}) = \dot{\theta}^T J^T J \dot{\theta}$ (Quadratic Form)

Level sets of this are ellipsoids.

Eigenvalues of J^TJ are all non negative

$\sigma_i = \sqrt{\lambda_i(J^TJ)}$ (sorted!)

Manipulability Measure.

1. σ_{min}, smallest singular value.

Aside: If $J \in \mathbb{R}^{6\times7}$ 7DOF manipulator take σ_i

$$J = U\Sigma V^T = \underbrace{\overset{6\times6}{[U]}}_{\text{orthogonal}} \begin{bmatrix} \sigma_1 & & 0 \\ & & 0 \\ & \sigma_6 & 0 \end{bmatrix} \underbrace{\overset{7\times7}{[V^T]}}_{\text{orthogonal}}$$

dimension of domain (θ space)

2. Inverse of condition # of J

$$\text{cond}(J) = \frac{\sigma_{max}}{\sigma_{min}}$$

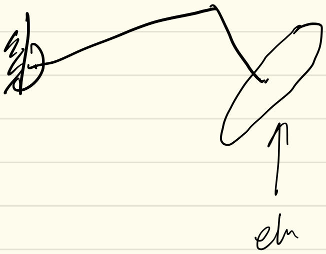
So we take $\frac{\sigma_{min}}{\sigma_{max}} = \frac{\sigma_6}{\sigma_1}$ as manipulability measure.

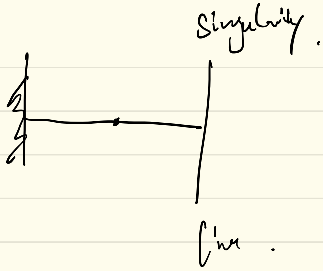
3. Det of J

→ only works if $n=6$

→ not great numerically

$\det J = \lambda_1 \cdots \lambda_n$ product of evals of J





① Clean up SVD

$J = U \Sigma V^T \in R^{6 \times n}$ $n > 6$ $\Sigma = [\underbrace{\bar{\Sigma}}_{6\times 6} \mid \underbrace{0}_{6 \times (n-6)}]$

U = eign vector of JJ^T, evalue are (σ_1^2, σ_6^2)

V = " " of J^TJ. e-value are $(\sigma_1^2, \cdots, \sigma_6^2, \underbrace{0, \cdots 0}_{n-6})$

U, V ortho-normal

Facts:

since JJ^T (J^TJ) is symmetric, it has orthogonal e-vectors.

since JJ^T (J^TJ) is positive, semi-definite $\Rightarrow$ eign values are all non-negative.

Note:

$$JJ^T = U\Sigma V^T V \Sigma U^T = U[\bar{\Sigma} \mid 0] \begin{bmatrix} \bar{\Sigma}^T \\ 0^T \end{bmatrix} U^T$$

$$= U\left[\bar{\Sigma}\bar{\Sigma}^T + OO^T\right]U^T$$

orthogonal.

$$= U\bar{\Sigma}^2 U^T$$

$$= U\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_6^2 \end{bmatrix}U^T$$

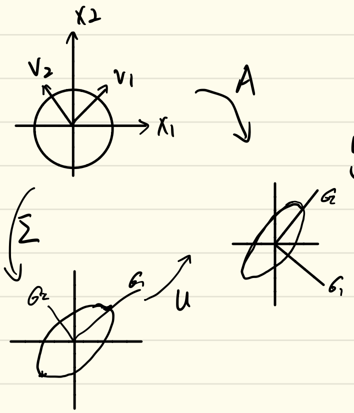
verify relation between SVD eigh structures

Likewise.

$$J^TJ = V\begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix}V^T$$

2x2 case

$y = Ax$, $x, y \in \mathbb{R}^2$, $\underset{2\times2}{A} = \underset{2\times2}{U}\,\underset{2\times2}{\Sigma}\,\underset{2\times2}{V^T}$



$$x = v_1a_1 + v_2a_2$$

$$y = Ax = U\Sigma V^T x$$

$$= U\Sigma\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}\left[a_1v_1 + a_2v_2\right] = \begin{pmatrix} a_1v_1v_1^T + a_2v_2v_2^T \\ a_1v_2^Tv_1 + a_2v_2v_2^T \end{pmatrix}$$

$$= U\Sigma\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= [u_1\ u_2]\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= [u_1\ u_2]\begin{bmatrix} \sigma_1 a_1 \\ \sigma_2 a_2 \end{bmatrix}$$

$$= \sigma_1a_1u_1 + \sigma_2a_2u_2$$

② Transposed Jacobian Control.

Recall $\dot{\theta} = u$ control system u = input $\dot{\theta}$ = joint velocities.

$$u = -k\,[J_{st}^{b}(\theta)]^{-1}\xi$$

where $\exp\{\hat{\xi}\} = g_{st^*}^{-1}\, g_{st}$

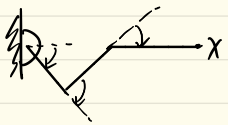
Turns out, we can also use $u = -k\,[J_{st}^{b}(\theta)]^{T}\xi$

Transpose, not inverse.

Motivation: Gradient-based control of a 3 DOF manipulator.

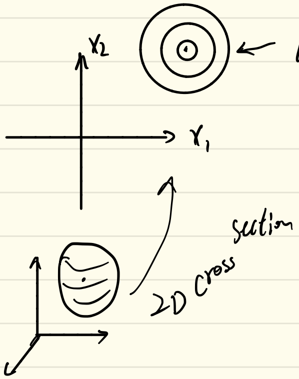
Let $x = f(\theta)$. $x \in \mathbb{R}^3$

↳ position of "tool"



Define error $V = \frac{1}{2}\|x - x^*\|^2$

gradient vector orthogonal to level sets.

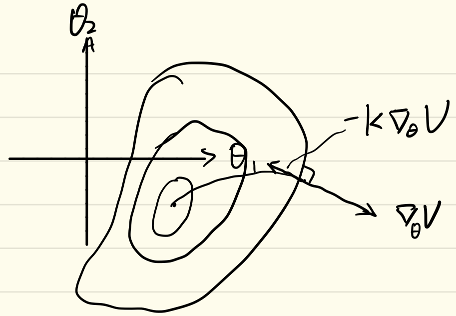


$$V(\theta) = \frac{1}{2}\|\underbrace{f(\theta)}_{x} - x^*\|^2$$

$$\dot{\theta} = -\nabla V(\theta)$$

Aside

$$\nabla V = \begin{bmatrix} \frac{\partial V}{\partial \theta_1} \\ \vdots \\ \frac{\partial V}{\partial \theta_n} \end{bmatrix} = \left[\frac{\partial V}{\partial \theta}\right]^T$$



$$\nabla V = \left(\frac{\partial V}{\partial \theta}\right)^T$$
$$= \left(\frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial \theta}\right)^T$$
$$= \left(\frac{\partial x}{\partial \theta}\right)^T \left(\frac{\partial V}{\partial x}\right)^T$$
$$= J^T \nabla_x V$$

$$\dot{\theta} = -k \nabla_\theta V$$
$$= -k J^T \nabla_x V$$

$$\nabla_x V = (x - x^*)$$

Inverse Kinematics

Given end-effector pose, find joint angles.

In Prob Set 4, You solved for θ_1, θ_2 when $x \in W$.

More general set of methods:

works for 6 DOF manipulators w/ spherical wrist & some others.

Break down inverse kinematics into so-called Paden-Kahan sub problems.

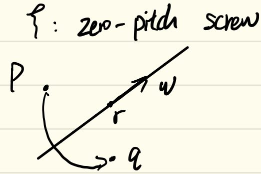
Three s.p. in MLS:

S.P. 1

Zero pitch

Given ξ_1, p_1, q

$$e^{\hat{\xi}\theta} p = q$$



$h=0$

Find θ: $\theta = \operatorname{atan2}(\omega^T(u' \times v'),\ u'^T v')$

$u' = u - \omega\omega^T u$ $\quad u = p - r$

$v' = v - \omega\omega^T v$ $\quad v = q - r$

solution exists if $\omega^T u = \omega^T v$, $\|u'\| = \|v'\|$

S.P. 2

Let ξ_1, ξ_2 be zero-pitch. unit magnitude twist with intersecting axes.

Find θ_1, θ_2, such that

$$e^{\hat{\xi}_1\theta} e^{\hat{\xi}_2\theta_2} p = q.$$

⋮

0, 1 or 2 solutions. (see book)

S.P.3

ξ = zero pitch. unit magnitude.

$p, q \in \mathbb{R}^3$

Find θ s.t $\|q - e^{\hat{\xi}\theta} p\| = \delta$. for some given δ

$\vdots$

again, 0. 1 or 2 solutions.

Dynamics & Mechanics.

$\downarrow$ $\qquad\qquad\qquad \downarrow$

$\dot{x} = f(x) \qquad\qquad F = m\ddot{x}$

In Robotics, we are often interested in dynamical systems and often such systems can be modeled via a set of 1^{st} order differential equations.

$\dot{x} = f(x, u) \quad . \quad x \in R^n, u \in R^p$

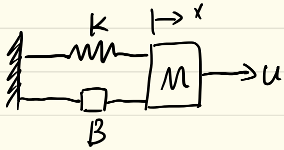
$\dot{x}_1 = f_1(x_1 \cdots x_n, u_1, \cdots u_p)$

$\vdots$

$\dot{x}_n = f_n(x_1 \cdots x_n, u_1 \cdots u_p)$

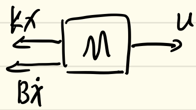
there. $x \in \mathbb{R}^n$ are the state variables, $u \in \mathbb{R}^p$ are the inputs.

Sometimes, particularly in mechanics problems, it is convenient to write a system of 2^{nd}-order ODE's. for example:



$x=0 \iff$ spring at rest

F.B.D.



$$m\ddot{x} = \Sigma F$$
$$= -kx - B\dot{x}$$

$$\Rightarrow M\ddot{x} + B\dot{x} + kx = u$$

Define state variables.

$$z = \begin{vmatrix} x \\ \dot{x} \end{vmatrix} \in \mathbb{R}^2 \quad \text{state variables.}$$

STATE: Minimum information at time t_0, such that, together will all future inputs, you can predict the evolution of the system.

Delay: Infinite state

$$u(t) \rightarrow \boxed{\text{Delay}} \rightarrow v(t) = u(t-T)$$

T = delay

Information needed (ie. state). to predict $u(t)$, $t \geq 0$
is $u(t)$, $t_0 - T \leq t < t_0$

Thus, together with $u(t)$. $t \geq t_0$

Lets we predict $x(t)$, $t \geq t_0$

$$\mathcal{L}\{u(t-T)\} = e^{-sT} U(s) \quad + \text{ I. C. term.}$$

$$e^{-sT} = \frac{e^{-sT/2}}{e^{sT/2}} = \frac{1 - sT/2}{1 + sT/2}$$

Back to $M\ddot{x} + B\dot{x} + kx = u$

$$z = \begin{vmatrix} x \\ \dot{x} \end{vmatrix} \in R^2$$

$$\dot{z} = \begin{vmatrix} \dot{x} \\ \ddot{x} \end{vmatrix} = \begin{vmatrix} z_2 \\ \frac{1}{m}(u - Bz_2 - kz_1) \end{vmatrix}$$

$$\ddot{x} = \frac{1}{m}(u - B\dot{x} - kx)$$

$$\begin{vmatrix} \dot{z}_1 \\ \dot{z}_2 \end{vmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{M} \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u$$

Any $w = Pz$

non-singular matrix.

would be OK choices for state variables.

Taxonomy

1. Linear vs Non-Linear.

$$\dot{x} = Ax + Bu = f(x, u)$$

linear - system

$$u_1(t) \to x_1(t)$$
$$u_2(t) \to x_2(t)$$

$$u \longrightarrow \boxed{\text{system}} \longrightarrow x$$

$$\alpha u_1(t) + \beta u_2(t) \to \alpha x_1(t) + \beta x_2(t)$$

set $u = 0$, $\dot{x} = Ax$

If $x_1(t)$, $x_2(t)$ are solutions. $\alpha x_1(t) + \beta x_2(t)$ is a solution.

$u = 0$ case $\quad x(t) = e^{At} x_0$

$f(x) = Ax$ is a linear function, ie. $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$

$$J = U\Sigma V^T = \sigma_1 u_1 v_1^T + \cancelto{0}{\sigma_2 u_2 v_2^T}$$

$$\Rightarrow \dot{x} = \sigma_1 u_1 \underbrace{v_1^T \dot{\theta}}_{\text{scalar}} = \text{scalar} \cdot u_1$$

$\text{Range}(J) = \text{span}\{u_1\}$

$\text{Ker}(J) = \text{span}\{v_2\}$

$$A = [u_1 \cdots u_n] \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_t & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$n-t$

$\mathrm{Range}(J) = \mathrm{span}\{u_1 \cdots u_t\}$

$\mathrm{Ker}(J) = \mathrm{span}\{v_{t+1} \cdots v_n\}$

Jacobian :

time-derivative of coords.

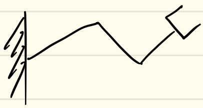
① Chain Rule Jacobian $\qquad \dot{x} = J\dot{\theta}$

$\underset{\#\text{ of DOF}}{\underline{\text{Coordinates}}}$ on both sides of a mapping

twist coordinate vector. $(V = (\dot{g}g^{-1})^\vee$ or $(g^{-1}\dot{g})^\vee)$

② Manipulator Jacobian. $\qquad V = J\dot{\theta}$

in SE(3) for pose . don't want to use Euler Angles.



$$g = \begin{bmatrix} c & -s & x \\ s & c & y \\ 0 & 0 & 1 \end{bmatrix}$$

relate $(\dot{x}, \dot{y}, \dot{\theta})$ & $(g^{-1}\dot{g})^\vee$

$$w^s = (\dot{R}R^T)^\vee = J \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

Taxonamy of systems

I. Linear vs. Non-linear.

$$\dot{x} = f(x, w), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^D$$

Linear: $f(x,u) = \underset{n \times n}{A}x + \underset{n \times D}{B}u$

$$u \Rightarrow \boxed{\text{system}} \longrightarrow x$$

$$u_1(t) \rightarrow \boxed{\text{sys}} \longrightarrow x_1$$

$$u_2(t) \rightarrow \boxed{\text{sys}} \longrightarrow x_2$$

$$\Rightarrow \alpha u_1(t) + \beta u_2(t) \longrightarrow \boxed{\text{sys.}} \Rightarrow \alpha x_1 + \beta x_2.$$

assuming $x(0) = x_0 = 0$

Aside: $x(t) = e^{At}x + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$

Non-linear: everything else.

$$\dot{x} = x^2 = f(x)$$

f is not linear in x.

II. Autonomus vs Non-autonomus

time-invariant time-variant.

Given a system Σ, and $x(t)$, $u(t)$ (state input) that solves Σ,

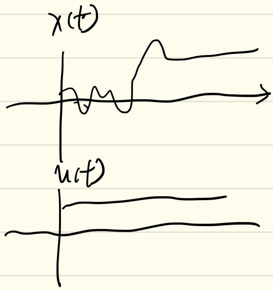
then $x(t-t_0)$ and $u(t-t_0)$ also solves Σ.

e.g. any system of the form $\Sigma: \dot{x} = f(x, u)$

Pf:

Let $x(t)$, $u(t)$ be a solution to Σ. let $\bar{x}(t) = x(t-t_0)$,

$\bar{u}(t) = u(t-t_0)$



If f explicitly depends on time,

Σ is time-varying

III Mechanical System.

Those that aside from Newton's laws!

We will focus on These. (typically Autonomus, nonlinear)

IV. ∞: lots of other -- of systems.

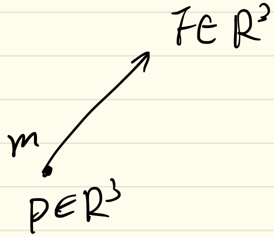
Dynamics of Rigid bodies

point masses:

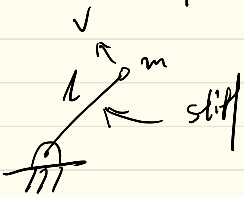
In an inertial frame, Newton's second law:

$$F = m\ddot{p} \quad , \quad F, p \in \mathbb{R}^3$$

where $F = \sum_i F_i$. sum of all forces acting on the point mass.



Example: Centripital Forces (inward-pointing forces)



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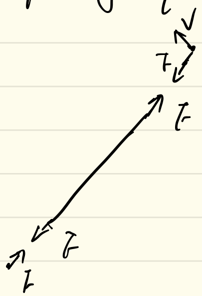
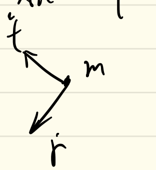


Attach a frame.



Decompose force

$F_t = 0$

$F_n = F = \text{unknown}.$

$$a_r = \frac{v^2}{l}$$

$$\Rightarrow F = m\frac{v^2}{l}$$

Real force! Centripetal Force.

No outward force on mass. there is one on string.

System of Rigidly connected points.

$$F_i = m_i \ddot{P}_2 \qquad F \cdot P \in \mathbb{R}^3 \quad , \quad i = 1, 2, \cdots n$$

Constraint

$$\| P_i - P_j \|^2 - l_{ij}^2 = 0$$

More generally

$$l_{ij}(p_1 \cdots p_n) = 0 \qquad l_{ij} : \mathbb{R}^3 \Rightarrow \mathbb{R}.$$

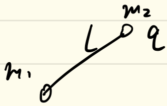
such a constraint (on position variables) is holonomic constraint.

Constraint forces are orthogonal to the constraints of the form.

$$\Gamma_j = \Big[\underbrace{\frac{\partial h_j}{\partial P_{i,1}}}_{x_1}, \underbrace{\frac{\partial h_j}{\partial P_{i,2}}}_{y_1}, \cdots \underbrace{\frac{\partial h_j}{\partial P_{n,3}}}_{z_n} \Big]^T = \nabla h_j.$$

Example:

Two masses, connected by a massless rod:



$$h(p, q) = \|p - q\|^2 - L^2$$

Constraint Forces are $\perp$ to the constraints.

So, if $\vec{x} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \in R^{3n}$

And $h_j : \mathbb{R}^{3n} \to R$, then.

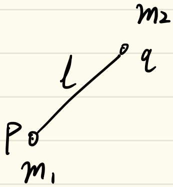
The constraint forces are of the form

$$F_j = \lambda_j \Gamma_j \in \mathbb{R}^{3n}$$

where $\Gamma_j = \nabla h_j = \begin{bmatrix} \frac{\partial x_j}{\partial x_1} \\ \vdots \\ \frac{\partial x_j}{\partial x_{3n}} \end{bmatrix}$

Example

Massless rod connecting two point masses.



$$h(p,q) = \|p-q\|^2 - l^2$$

$$= (p-q)^T(p-q) - l^2$$

$$\Rightarrow \Gamma = \nabla h = [\underbrace{\frac{\partial h}{\partial p}}_{1\times 3}, \underbrace{\frac{\partial h}{\partial q}}_{1\times 3}]^T \in \mathbb{R}^6$$

$$\frac{\partial h}{\partial p} = 2(p-q), \quad \frac{\partial h}{\partial q} = 2(q-p)$$

$$\Gamma = 2\begin{bmatrix} p-q \\ q-p \end{bmatrix}$$

$p-q$ ← direction of force on 1^{st} particle

$q-p$ ← direction of force on 2^{nd} particle.

Back to n-point case.

$$h_j(p_1 \cdots p_n) \overset{(4.2)}{=} 0, \quad j = 1 \cdots k \quad \text{. constraints}$$

$$F_i = m_i \ddot{p}_i \qquad \text{Newton's law.}$$

Stack up:

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \begin{bmatrix} m_1 I & & & \\ & m_2 I & & \\ & & \ddots & \\ & & & m_n I \end{bmatrix} \begin{bmatrix} \ddot{p}_1 \\ \vdots \\ \ddot{p}_n \end{bmatrix}$$

Break up

$F_{all} = F_{ext} + F_c$

$\uparrow$ constraint forces.

$$F_{ext} = \begin{bmatrix} m_1 I & & & \\ & \ddots & & \\ & & \ddots & \\ & & & m_n I \end{bmatrix} \begin{bmatrix} \ddot{P}_1 \\ \vdots \\ \vdots \\ \ddot{P}_n \end{bmatrix} + \sum_{j=1}^{k} \lambda_j \Gamma_j \quad \langle 4.3 \rangle$$

$3n$ equations.

(Where $F_c = -\sum_{j=1}^{k} \lambda_j \Gamma_j$)

$\Gamma_j = \nabla h_j$

$\lambda_1, \ldots \lambda_k$ are called Lagarange Multipliers.

We have $3n + k$ unknowns

$P_1 \cdots P_n$ $\quad$ $\lambda_1, \cdots \lambda_k$

$3n + k$ equations.

$\ddot{p}$ equations $\quad$ $h_j = 0$ equations

Better idea!

Let $f_i(\theta_i, \cdots \theta_m) \in \mathbb{R}^3$. $i = 1 \cdots n$ ← # points

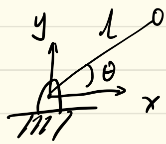
be known functions and suppose

$$P_i = f_i(\theta_1, \cdots \theta_m) \iff h_j(P_1, \cdots P_n) = 0$$
$$i = 1 \cdots n) \qquad j = 1, \cdots k$$

where $m = 3n - k$

$\theta_1 \cdots \theta_m$ are generalized cordinates and are unconstrainted.

Example:



$$P = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l\cos\theta \\ l\sin\theta \end{bmatrix} =: f(\theta)$$
$$h(P) = x^2 + y^2 - l^2$$

Note that $\forall \theta \in \mathbb{R} \mod 2\pi$

$$h(f(\theta)) = l^2(c^2 + s^2) = l^2 = 0$$

showed

$$P = f(\theta) \Rightarrow h(P) = 0$$

To show " $\Leftarrow$ "

Assume $h(p) = 0 \Rightarrow x^2 + y^2 - l^2 = 0$

then $\theta = \mathrm{atan2}(y, x)$ gives an angle such that $p = f(\theta)$

Read Principle of least Action by Feynman

Recap

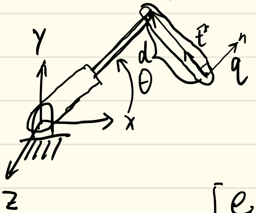
$$P_i = f_i(\theta_1, \dots \theta_m) \qquad i = 1 \dots n$$

$$m = 3n - k$$

$$\Updownarrow$$

$$h_j(P_1 \dots P_n) = 0 \quad , \quad j = 1, \dots k$$

Generalized Coordinates: $\theta_1, \dots \theta_m$



$P \in R^3$

$$h_1(P) = e^{3^T} P \Rightarrow P \in \pi_{xy}$$

$$\underbrace{h_2(P) = (P-q)^T n}_{P \in \pi_1} \quad , \quad \text{with } q^T e_3 = 0 \quad , \quad n = \text{const} \quad . \quad q = \text{const}.$$

$$h(p) = \begin{bmatrix} e_3^T p \\ n^T (p-q) \end{bmatrix}$$

$$\frac{\partial h}{\partial P} = \begin{bmatrix} e_3^T \\ n^T \end{bmatrix} \qquad e_3 \neq \alpha n$$

∞ choices of coordinates

① θ , with f st. $f(\theta)$

② t such that be $\perp e_3 . \perp n$

$$\hat{t} = \frac{n \times e_3}{\| n \times e_3 \|}$$

Consider a generalized coordinate. d.

$$\bar{f}(d) = q + \hat{t} d.$$

" $\Rightarrow$ "

$$h(\bar{f}(d)) = \begin{bmatrix} e_3^T q + \hat{t} d \\ [q + \hat{t} d]^T n \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{e_3^T q}^{0} + \cancel{e_3^T \hat{t} d}^{0} \\ \underbrace{\cancel{(\hat{t} d)^T n}^{0}}_{\hat{t} \perp n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

" $\Leftarrow$ " Assume $d(p) = 0$

Let $d = (p - q)^T \hat{t}$

Check if $\bar{f}(d) = p$:

$$\bar{f}(d) = q + \hat{t} (p - q)^T \hat{t}$$

Principle of least action

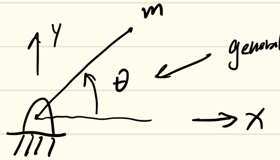
Let $\theta_1, \cdots, \theta_n$ be generalized coordinates. write $\theta \in \mathbb{R}^n$

Define kinetic energy and potential energy

$T(\theta, \dot{\theta}) \qquad V(\theta)$

e.g.

Pendulum



$f(\theta) = l \begin{bmatrix} c\theta \\ s\theta \end{bmatrix}$

$$T = \frac{1}{2} m \|\dot{p}\|^2 = \frac{1}{2} m \left\| \frac{d}{dt} f(\theta) \right\|^2 = \frac{1}{2} m \left\| \dot{\theta} \begin{bmatrix} -l s\theta \\ c\theta \end{bmatrix} \right\|^2$$

$$= \frac{1}{2} l^2 m \dot{\theta}^2$$

$V = mgl\sin\theta$

Define Lagrangian

$$L(\theta, \dot{\theta}) = T(\theta, \dot{\theta}) - V(\theta)$$

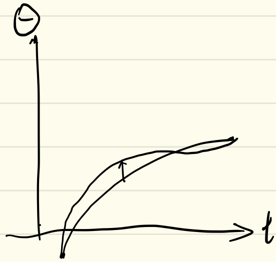
And the action

$$S = \int_{t_0}^{t_f} L(\theta, \dot{\theta}) dt$$

Principle of least action: the trajectory of system minimize S

$$\Rightarrow \quad \delta \int L(\theta, \dot{\theta}) dt = 0$$

variational derivative.



$$\delta \int L(\theta, \dot{\theta}) \, dt = 0$$

$$= \int \left(\frac{\partial L}{\partial \theta} \partial \theta + \frac{\partial L}{\partial \dot{\theta}} \partial \dot{\theta} \right) dt$$

$$= \int \left(\frac{\partial L}{\partial \dot{\theta}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \right) \partial \theta \, dt = 0$$

$$\Downarrow$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0$$

$$S = \int L \, dt$$

$$\delta S = \int \left(\frac{dL}{d\theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \right) \delta \theta \, dt = 0$$

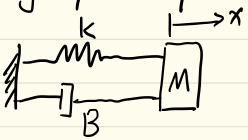
Generating n DOF

$$\frac{\partial L}{\partial \theta_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} = 0 \quad \text{(no external forces)}$$

If we add external forces:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial}{\partial \theta_i} = \Upsilon_i \quad \text{(upsilon)}$$

Obligatory Example



$$T(x,\dot{x}) = \frac{1}{2}m\dot{x}^2$$

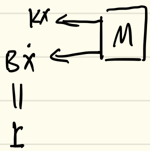
$$V(x) = \frac{1}{2}kx^2 \quad \text{(Hookean Spring)}$$

$$L(x,\dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + kx = \Upsilon$$

Look at F.B.D to figure

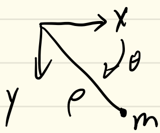


$$\Rightarrow \quad M\ddot{x} + kx = -B\dot{x}$$

$$M\ddot{x} + kx + B\dot{x} = 0$$

Specific equations you get depend on what you choose for generalized Coordinates, but what the system does does not!

Example.



Two choices:

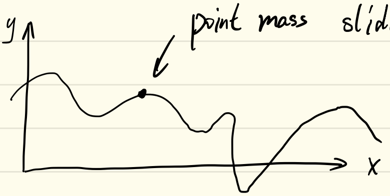
$$\begin{vmatrix} x \\ y \end{vmatrix}$$

$$\begin{vmatrix} \rho \\ \theta \end{vmatrix}$$

$$\begin{tikzcd}
(x, y) \arrow[rr, leftrightarrow, "\text{Change variables}"] \arrow[d] & & (\rho, \theta) \arrow[d] \\
\text{Lagrange's Equations} \arrow[rr, leftrightarrow, "\text{Change varibles}"] & & \text{Wired Lagrange's Equations.}
\end{tikzcd}$$

Example

point mass sliding on mass.



Generalized coordinate choices.

$$\theta = x$$

Another choice : $\theta = s$. s = distance along curve.

Let $\theta = x$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$= \frac{1}{2} m \left[\dot{x}^2 + (f'(x)\dot{x})^2 \right] + mg f(x)$$

$$\frac{\partial L}{\partial x} = m \left(f(x)\dot{x} \right) f''(x) \quad - mg f'(x)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{d}{dt} \left(m\dot{x} + m f'(x)\dot{x} f'(x) \right)$$

$$= m\ddot{x} + m\ddot{x} f'^2(x) + 2m f'(x) \dot{x} f''(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

conservative

$$m(1+f'(x)^2)\ddot{x} + mf'(x)f''(x)\dot{x}^2 + mgf'(x)$$

$$m\ddot{x} + B\dot{x} + kx = 0$$

dispated

Lyapunov Stability ($\dot{q}$ Control)

Consider a time-variant system

$$\dot{x} = f(x) \quad (\text{not } f(x,t))$$

If $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + N(q) = 0$

$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ and get it in form

Can make this a control problem by introducing an INPUT:

$$\dot{x} = f(x, u)$$

Many (many!) definitions of stability.

- Input-to-state stability

  Roughly. finite input leads to finite state values.

- Input-output stability

  Roughly if we define

  say $y = y(t)$

- Linear Stability

  Stability of linearization around an equilibrium.

- Lyapunov Stability.

(*) Consider a system $\dot{x} = f(x)$, $x(0) = x_0$, $x \in \mathbb{R}^n$

Suppose $x^* = 0$ is an equilibrium.

(If equilibrium is not 0, change ~~stability~~ coordinate)

Equilibrium:

$f(x^*) = 0$

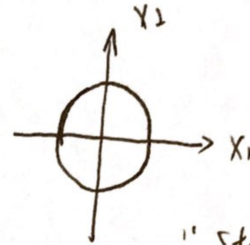
Definition: Stability

(In the sense of Lyapunov)

$\delta(\varepsilon)$

The equilibrium $x^* = 0$ of $(*)$ is stable if for any $\varepsilon > 0$, $\exists \delta > 0$

s.t. for all $\|x_0\| < \delta$, $\|x(t)\| < \varepsilon \ \forall t \geq 0$



"Start close $\Downarrow$ Stay close"

Definition: Asymtotic Stability

$x^* = 0$ is asymtotically stable if:

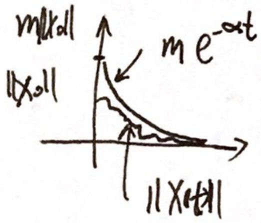
1. $x^* = 0$ is stable in the sense of Lyapunov
2. $\exists \delta > 0$ s.t. $\|x_0\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$

Definition: Exponential Stability.

The equilibrium $x^* = 0$ is exponentially stable if:

$\exists m, \alpha, \varepsilon > 0$, s.t. $\|x_0\| < \varepsilon \Rightarrow \|x(t)\| < m e^{-\alpha t} \|x_0\|$

The largest ~~constai~~ constant α is called the convergence rate.



Example of a system that is

1. Stable (Not Asymtotially)

$\ddot{y} + y = 0$

State space: $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$

Solutions:

$y(t) = A\sin(t+\phi)$ (A, ϕ depend on $y(0), \dot{y}(0)$)

$$\Rightarrow X(t) = \begin{bmatrix} A\sin(t+\phi) \\ A\cos(t+\phi) \end{bmatrix}$$

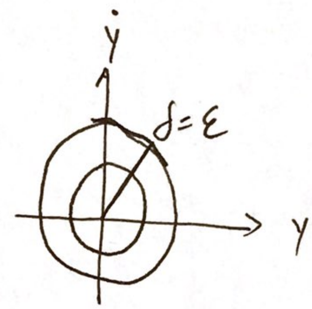
$\Rightarrow \|X(t)\| = |A|$ for all t.

Stability:

For all $\varepsilon > 0$, let $\delta = \varepsilon$

Now, if $\|X_0\| = |A| < \delta$

$\Rightarrow \|X(t)\| = |A| < \varepsilon$

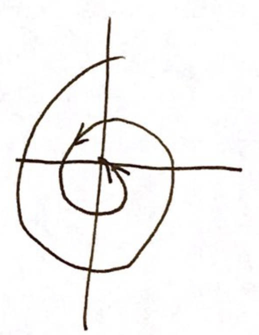


II) Asymtotically stable:

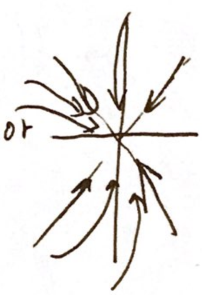
$$m\ddot{x} + b\dot{y} + ky = 0$$

$$X = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{X} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{M} \end{bmatrix}}^{A} X$$

Assume $m, k, b > 0$



or



"Easy" to show:

$$\lim_{t\to\infty} X(t) = 0$$

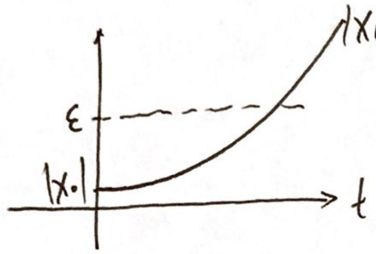
Also "Easy" to show:

Stable (in the sense of Lyapunov)

☐ unstable

$\dot{x} = ax$, $a > 0$, $x \in \mathbb{R}$

$\Rightarrow x(t) = x_0 e^{at}$

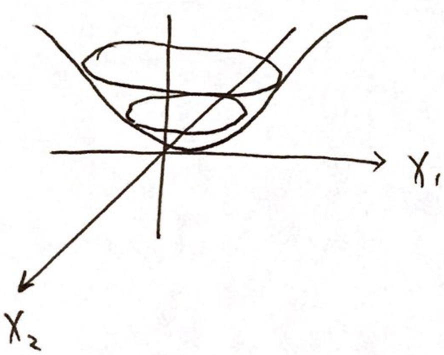


Definition : A continuous function $V: \mathbb{R}^n \to \mathbb{R}$ is locally positive-definite

if for some $\varepsilon > 0$; for some strictly increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}$.

s.t $V(0) = 0$

$V(x) \geq \alpha(\|x\|)$, $\forall x \in B_\varepsilon = \{ x \in \mathbb{R}^n : |x| < \varepsilon \}$



If $V(x)$ is locally definite and $\dot{V} \leq 0$.

(neg definite)

$\Rightarrow$ stability